

**General Equilibrium Theory
in Infinite Dimensions:**
An Application of Fredholm Index Theory

Enrique Covarrubias

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Abstract

This thesis deals with generic determinacy of equilibria for infinite dimensional economies. Our work could be seen as an infinite-dimensional analogue of Dierker and Dierker (1972), by characterising equilibria of an economy as a zero of the aggregate excess demand, and studying its transversality. In this case, we can use extensions of the Sard-Smale theorem. Assuming separable utilities, we give a new proof of generic determinacy of equilibria. We define regular price systems in this setting and show that an economy is regular if and only if its associated excess demand function only has regular equilibrium prices. We also define the infinite equilibrium manifold and show that it has the structure of a Banach manifold.

We also provide conditions that guarantee global uniqueness of equilibria for smooth infinite economies. We do this by introducing to the economic literature the notion of Z-Rothe vector fields that will allow us to construct an index theorem à la Dierker (1972); this shows that the number of equilibria is odd and in particular gives a new proof of existence.

Extending the finite dimensional results of Balasko (1988), we characterise the equilibrium manifold as a covering space of the set of economies and we study global conditions under which the natural projection map is a diffeomorphism. We furthermore study the effects that critical equilibria have on the global invertibility of the projection map.

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Chapter 1

Introduction

This work studies questions of determinacy and the number of equilibria within the differential approach to the theory of general equilibrium (GE) in infinite dimensions. It is organised as follows.

Chapter 2 is an introduction to the differential approach to the theory of general equilibrium. This theory is a rich and active area of research for which providing a full review is a task outside the scope of our work. We have only attempted to review aspects of determinacy and the number of equilibria for finite dimensions. We then explain results that have been obtained by several authors in the search to generalise this theory to infinite dimensions. We explain some of its successes and some of the shortcomings.

The new results obtained in this thesis are presented in chapters 3 to 5. In chapter 3 we present a study of determinacy in infinite dimensions. While results in this area have been studied thoroughly over the years, we present an original

contribution by using a Fredholm-index approach to general equilibrium. Our approach consists in considering a family of functions (excess demand functions) parametrised by elements in a Banach manifold (the space of initial endowments). This new setting is the right one to study parametric transversality and applying the Smale-Sard theorem. Among other results we show that equilibria is “generically” determined.

Knowing that equilibria are locally unique, we aim in chapter 4 to count them. We attack this problem by generalising to infinite dimensions a classical work of Dierker (1972). We show that the excess demand function is a Z-Rothe vector field. This is a new structure of vector fields on Banach manifolds that has never appeared in the economic literature. The relevance of this structure is that it allows us to construct an index theorem where we show that the sum of the indices of the zeros of this Z-Rothe vector field equals 1. The zeros of this vector field are exactly the equilibria of the economy. Among the implications of our index theorem is that equilibria will always exist and there will be an odd number of them. More importantly, it will give us conditions that guarantee global (not just local) uniqueness of equilibria.

In chapter 5 we further study global uniqueness of equilibria. We propose a “framework” that should work for a wide range of applications within general equilibrium theory and we apply it to three situations: finite dimensions with complete and incomplete markets, and infinite dimensions. We give precise conditions on both the excess demand function and on the natural projection map that are equivalent to having global uniqueness of equilibria. Exploiting the Fred-

holm setting, we give a topological characterisation of the equilibrium set, as an infinite dimensional analogue to Balasko (1988). We finally give an indication on how the existence of critical equilibria affects global uniqueness. The idea is that outside the set of critical equilibria the projection map from the equilibrium manifold to the space of parameters is a covering space map.

We finally provide for reference an appendix collecting some facts from linear and nonlinear functional analysis.

Chapter 2

Equilibria: from finite to infinite dimensions

2.1 Introduction

In this chapter we review the differential approach to general equilibrium theory in finite dimensions and we explain how to extend it to an infinite-dimensional setting. There are several books and papers reviewing the finite-dimensional theory that could probably be encompassed in three bibliographical categories. The first category is the “classical” theory that includes Arrow and Hahn (1986) and Debreu (1959). The more modern, fundamental and “differential” approach is reviewed in Balasko (1988), Debreu (1982, 1983), Dierker (1974, 1982) and Mas-Colell (1985). A third category may include the many more extensions of general equilibrium theory some of which are available throughout the four volumes of the *Handbook of Mathematical Economics*. We will only concentrate on the relevant results regarding determinacy and the number of equilibria.

We begin in section 2.2 by setting the economic framework in its finite-dimensional setting. The main issue brought up is the definition of equilibrium. In section 2.3 we study the determinacy (i.e., local uniqueness) of equilibria. There are three authors who have independently addressed this issue: Debreu (1970), E. Dierker and H. Dierker (1972) and Balasko (1975b, 1988).

In section 2.4 we study the number of equilibria. This question was answered originally by Dierker (1972) by constructing an “index theorem” à la Poincaré-Hopf. The main result is that, generically, the number of equilibria will be odd. This index theorem can also be used to determine necessary and sufficient conditions for global uniqueness of equilibria. The number of equilibria was rediscovered, independently, by Balasko (1975b, 1988) using a different approach involving the degree of maps.

Finally in section 2.5 we review some of the known results when extending general equilibrium to an infinite-dimensional setting. The ultimate goal of the dissertation is to extend the results of sections 2.2-2.4 to infinite dimensions.

2.2 The Economic Framework

2.2.1 Exchange economies

We consider an economy with a finite number I of agents. We also consider a finite number L of commodities so that the commodity space is a subset of \mathbb{R}^L .

We assume throughout that the **consumption space** X is

$$X = \mathbb{R}_{++}^L = \{x = (x_1, \dots, x_j, \dots, x_L) \in \mathbb{R}^L : x_j > 0, \forall j\},$$

the positive cone of \mathbb{R}^L . The **price space** S is

$$S = \left\{ P = (P_1, \dots, P_L) \in \mathbb{R}_{++}^L : \|P\| = \left(\sum_{j=1}^L P_j^2 \right)^{\frac{1}{2}} = 1 \right\},$$

the positive orthant of the unit sphere.

An **exchange economy** is parametrized for each agent $i = 1, \dots, I$ by their **initial endowments** $\omega_i \in X$ and their **individual demand functions**

$$f_i : S \times (0, \infty) \rightarrow X.$$

Each agent can calculate his wealth $w_i \in (0, \infty)$ by setting

$$w_i = P \cdot \omega_i := \langle P, \omega_i \rangle$$

The maps $f_i(P, w_i)$ are solutions to the optimization problem

$$\max_{P \cdot x = w_i} u_i(x)$$

where we assume that the **utility functions** $u_i(x) : X \rightarrow \mathbb{R}$ satisfy for each $i = 1, \dots, I$ the following:

1. u_i is of class C^r for $r \geq 2$;
2. $Du_i(x) \in \mathbb{R}_{++}^L$ (strict monotonicity);

3. $h^T D^2 u_i(x) h < 0$ for all $h \neq 0 : h \cdot Du_i(x) = 0$ (differentiable strict convexity);
4. $\{x \in \mathbb{R}_{++}^L : u_i(x) \geq u_i(\bar{x})\}$ is closed in \mathbb{R}^L for all $\bar{x} \in \mathbb{R}_{++}^L$.

It is known that the maps $f_i : S \times (0, \infty) \rightarrow X$ are of class C^1 for each $i = 1, \dots, I$ and are supposed to satisfy:

$$P \cdot f_i(P, w) = w \quad \text{for any } P \in S \quad \text{and for any } w \in (0, \infty) \quad (2.1)$$

In this dissertation we assume that the utility functions are fixed, so that the only parameters defining an economy are the initial endowments.

2.2.2 Excess demand functions

Let $\omega = (\omega_1, \dots, \omega_I) \in \Omega = X^I$. For a fixed economy $\omega \in \Omega$ the **aggregate excess demand function** is a map $Z_\omega : S \rightarrow \mathbb{R}^{L-1}$ defined by deleting the last component of

$$Z_\omega(P) = \sum_{i=1}^I (f_i(P, \langle P, \omega_i \rangle) - \omega_i)$$

Since we have normalized prices so as to be in S , the positive orthant of the unit sphere, and given that $P \cdot Z_\omega(P) = 0$ for all $P \in S$, then we can interpret Z_ω as a vector field tangent to S .

We also define $Z : \Omega \times S \rightarrow \mathbb{R}^{L-1}$ by the evaluation

$$Z(\omega, P) = Z_\omega(P)$$

2.2.3 Equilibrium set

The most important definition is that of an equilibrium. We follow closely the notation of Balasko (1988).

Definition 1. We say that $P \in S$ is an **equilibrium** of the economy $\omega \in \Omega$ if $Z_\omega(P) = 0$. We denote the **equilibrium set**

$$\Gamma = \{(\omega, P) \in \Omega \times S : Z(\omega, P) = 0\}.$$

We could also talk about the Walras correspondence $W : \Omega \rightarrow S$ that assigns to each economy $\omega \in \Omega$ its corresponding equilibria. General equilibrium could be seen as the study of this correspondence.

2.3 Determinacy of Equilibria

2.3.1 Debreu's approach: Sard's theorem

Debreu (1970) began studying the notion of determinacy of equilibria. In a very simplified manner, the idea is as follows. Arrow and Debreu (1954) have shown that for each economy, that is for each parameter $\omega \in \Omega$, an equilibrium exists. Ideally, we would like the model to have a unique solution; we know now that this is not the case, but we would like to see if at least “generically” they are finite and explore suitable conditions that guarantee uniqueness.

The mathematical tool used for studying determinacy turned out to be Sard's theorem. By now this is a classical result and can be found for instance in the

books of Milnor (1965) or Hirsch (1976).

Recall that if F is a continuously differentiable function from an open subset U of \mathbb{R}^a to \mathbb{R}^b , the set $C = \{x \in U : \text{rank } dF_x < b\}$ is the set of *critical points*, $F(C)$ is the set of *critical values* and a value that is not critical is called a *regular value*.

Also recall that having Lebesgue measure zero means that given any $\epsilon > 0$, it is possible to cover the set of critical values by a collection of cubes in \mathbb{R}^b giving total b -dimensional volume less than ϵ . We also say that a subset of \mathbb{R}^b is *null* if it has Lebesgue measure zero in \mathbb{R}^b . Sard's theorem will tell us the “size” of critical values.

Theorem 2. (*Sard's theorem*) *Let F be a smooth map from an open set U of \mathbb{R}^a to \mathbb{R}^b . Then the set of critical values has Lebesgue measure zero.*

There is also a stronger version.

Theorem 3. (*Sard's theorem*) *Let F be a continuously differentiable function from an open subset U of \mathbb{R}^a to \mathbb{R}^b . If all the partial derivatives of F to the c -th order included, where $c > \max(0, a - b)$, exist and are continuous, then the set of critical values of F has Lebesgue measure zero in \mathbb{R}^b .*

Debreu (1970) required the *desirability assumption* below which expresses the idea that every commodity is desired by the i th consumer.

Assumption 4. (Desirability assumption)

If the sequence $(P^q, w_i^q) \in S \times (0, \infty)$ converges to $(P^0, w_i^0) \in \partial S \times (0, \infty)$, then $\|f_i(P^q, w_i^q)\|$ tends to ∞ .

The classical result of Debreu is as follows.

Theorem 5. (Debreu, 1970) *Given I continuously differentiable individual demand functions (f_1, \dots, f_I) , if some f_i satisfies the desirability Assumption 4, then the set of $\omega \in \Omega$ that have an infinite number of equilibria, has closure with Lebesgue measure zero.*

Additionally, Debreu shows how the set of equilibria depend on the parameters ω . Define the map

$$F : S \times (0, \infty) \times X^{I-1} \rightarrow \mathbb{R}^{LI}$$

that for each generic element $(P, w_1, w_2, \dots, w_I)$ assigns the value $(\omega_1, \dots, \omega_I)$ where (w_2, \dots, w_I) is unchanged while w_1 is given by

$$w_1 = f_1(P, w_1) + \sum_{i=2}^I f_i(P, P \cdot \omega_i) - \sum_{i=2}^I \omega_i$$

Debreu shows the following:

Theorem 6. (Debreu, 1970) *Under the assumptions of Theorem 5, if $\omega^0 \in \Omega$ is a regular value of F , there are an open neighborhood V of ω^0 and k continuously differentiable functions g_1, \dots, g_k from V to S such that for every ω in V , the set of equilibria of ω consists of the k distinct elements $g_1(\omega), \dots, g_k(\omega)$.*

In particular, Theorem 6 shows that if $\omega \in \Omega$ is a regular value of F and $r(\omega)$ denotes the number of elements of $W(\omega)$, the function r from regular points to

the set of non-negative integers is *locally constant*.

2.3.2 Regular economies and regular equilibria

There is another approach to the study of determinacy due to Balasko (1975, 1988) that becomes clear by studying the structure of the equilibrium set.

First, to give a manifold structure to the equilibrium set $\Gamma \subset \Omega \times S$, consider the map $Z : \Omega \times S \rightarrow \mathbb{R}^{L-1}$. One can see (Balasko 1975b, 1988) that DZ is surjective, which in particular implies that 0 is a regular value of Z . Hence, the set of equilibria $\Gamma = Z^{-1}(0)$ is a differentiable manifold of dimension LI , called the **equilibrium manifold**.

Now, consider the projection map $\pi : \Gamma \rightarrow \Omega$. We say that an economy $\omega \in \Omega$ is **regular** (resp. **critical**) iff ω is a regular (resp. critical) value of the projection map $\pi : \Gamma \rightarrow \Omega$.

Similarly, we say that an equilibrium $(\omega, p) \in \Gamma$ is **critical** if it is a critical point of the projection map π . Otherwise, the equilibrium is called **regular**.

With these notions one can see that the economy $\omega \in \Omega$ is regular iff all equilibrium prices of ω are regular (cf. Balasko 1975b, 1988).

Since we will be extending Balasko's previous result to infinite dimensions,

at this stage it is worth mentioning the transversal density theorem. The idea behind this result is the following: in the context of representations of manifolds with suitable hypothesis, almost every point in the domain is represented by a transversal map (cf. Abraham-Robbin, 1967).

Theorem 7. (*Transversal Density Theorem*) Let A, M, N be C^r manifolds, $\rho : A \rightarrow C^r(M, N)$ a C^r representation, $W \subset N$ a submanifold and $ev_\rho : A \times M \rightarrow N$ the evaluation map. Define $A_W \subset A$ by

$$A_W = \{a \in A : \rho_a \pitchfork W\}$$

Assume that:

1. M has finite dimension n and W has finite codimension q in N
2. A and M are second countable
3. $r > \max(0, n - q)$
4. $ev_\rho \pitchfork W$

Then A_W is residual (and hence dense) in A .

Notice that Ω, S and \mathbb{R}^{L-1} are C^r manifolds, $\hat{Z} : \Omega \rightarrow C^r(S, \mathbb{R}^{L-1})$ is a C^r representation and $\hat{Z} : \Omega \times S \rightarrow \mathbb{R}^{L-1}$ is the evaluation map. If we define $\Omega_{T_0S} = \{\omega \in \Omega : Z \pitchfork T_0S \subset TS\}$, then the transversal density theorem implies that the set of regular economies is dense in Ω .

2.4 The number of equilibria

In the previous section it was established that, generically, an economy has a finite number of equilibria. Can we then actually count them? This section aims to answer this question by showing that generically there will be an odd number of them. We begin in section 2.4.1 by reviewing some of the topological tools required, the most important of which is Hopf's lemma. Most of this material can be found in (Milnor, 1965) or (Hirsch, 1976). In section 2.4.2 we review the index theorem of regular economies.

2.4.1 Topological preliminaries

Let U be an open subset of \mathbb{R}^m and let $v : U \rightarrow \mathbb{R}^m$ be a smooth vector field. Suppose that inside U there is a unique isolated zero $z \in U$. We define the **index** ι of v at z to be the degree of the map

$$\bar{v}(x) = \frac{v(x)}{\|v(x)\|}.$$

The map \bar{v} maps a small sphere centered at z to the unit sphere. It is also worth mentioning that the concept of index is invariant under diffeomorphisms of U .

A classical result of differential topology is the Poincaré-Hopf Theorem. It relates indices of vector fields with the topology of the underlying manifold. Let $H_i(M)$ denote the i -th homology group of a manifold M .

Theorem 8. (*Poincaré-Hopf Theorem*)(cf. Milnor, 1965) *Let M be a compact*

manifold and v a smooth vector field on M with isolated zeros. If M has boundary, then v is required to point outward at all boundary points. The sum $\sum \iota$ of the indices at the zeros of such a vector field is equal to the Euler number

$$\chi(M) = \sum_{i=0}^m (-1)^i \operatorname{rank} H_i(M).$$

There are weaker versions of this result, for instance for compact domains of Euclidean space. Let $X \subset \mathbb{R}^m$ be a compact m -manifold with boundary. The **Gauss mapping**

$$g : \partial X \rightarrow S^{m-1}$$

assigns to each $x \in \partial X$ the outward unit normal vector at x .

Lemma 1. (*Hopf's lemma*) (cf. Milnor, 1965) *If $v : X \rightarrow \mathbb{R}^m$ is a smooth vector field with isolated zeros, and if v points out of X along the boundary, then the index sum $\sum \iota$ is equal to the degree of the Gauss mapping from ∂X to S^{m-1} . In particular, $\sum \iota$ does not depend on the choice of v .*

So if for instance X is the disk $\mathbb{D}^m = \{x \in \mathbb{R}^m : \|x\| \leq 1\}$, and the vector field points outward along the boundary, then $\sum \iota = 1$.

Indices of vector fields can be computed directly in terms of derivatives. The two lemmas below will show us how this is done. Consider a vector field v on an open set $U \subset \mathbb{R}^m$ and think of v as a mapping $U \rightarrow \mathbb{R}^m$, so that $dv_z : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined.

Lemma 2. *(cf. Milnor, 1965) The index of v at a non-degenerate zero z is either $+1$ or -1 according as the determinant of dv_z is positive or negative.*

Notice that if v preserves orientation, lemma 2 is telling us that v can be deformed in a neighbourhood of the zero smoothly into the identity without introducing new zeros. Hence the index is $+1$. Similarly, if v reverses orientation, v can be deformed into a reflection so that now its index is -1 .

Now, consider a zero z of a vector field w on an oriented manifold $M \subset \mathbb{R}^k$. Think of w as a map from M to \mathbb{R}^k so that the derivative $dw_z : TM_z \rightarrow \mathbb{R}^k$.

Lemma 3. *(cf. Milnor, 1965) The derivative dw_z actually carries TM_z into the subspace $TM_z \subset \mathbb{R}^k$, and hence can be considered as a linear transformation from TM_z to itself. If this linear transformation has determinant $D \neq 0$ then z is an isolated zero of w with index equal to $+1$ or -1 according as D is positive or negative.*

2.4.2 The index theorem

Since Debreu (1970) showed that equilibria are locally unique, we may ask if there is a way of counting them. This question was answered by Dierker (1972).

First recall that if $\zeta : S \rightarrow \mathbb{R}^{L-1}$ is the excess demand of economy $\omega \in \Omega$, a price system $P \in S$ is a regular equilibrium price system iff $\zeta(P) = 0$ and $[(\partial\zeta_1, \dots, \partial\zeta_{L-1}/\partial P_1, \dots, \partial P_{L-1})(P)]$ has full rank.

Definition 9. Define the **index** of P as

$$\text{index}(P) = \text{sign} \det \left[\frac{-\partial \zeta_1, \dots, \partial \zeta_{L-1}}{\partial P_1, \dots, \partial P_{L-1}}(P) \right]$$

Dierker (1972) chose the sign so that locally stable equilibria have index $+1$. The local stability that he refers to is a price adjustment process $\dot{p} = v(p)$ for an economy ω where $v : S \rightarrow \mathbb{R}^{L-1}$ is continuously differentiable, $v(p) = 0$ iff $\zeta(P) = 0$ and $g := \text{id} + v$ satisfies that there is a homotopy $g_t : S \rightarrow \mathbb{R}^{L-1}$, $0 \leq t \leq 1$ between $g_0 := g$ and the constant mapping g_1 , defined by $g_1(p) := (1/L, \dots, 1/L) \in \mathbb{R}^{L-1}$ for all $p \in S$, such that $\bigcup_{t=0}^1 \{p \in S : g_t(p) = p\}$ is compact.

Now, from Hopf's lemma, we know that the indices of zeros of vector fields on compact manifolds add up to a topological invariant: the Euler characteristic, $\chi(M)$, of M . In our setting, the price space is not compact but the desirability assumption assures a nice behaviour around the boundary that guarantees that the index sum still holds. The index theorem is the following:

Theorem 10. *(Dierker, 1972) For any regular economy, we have*

$$\sum_{\{P \in S : \zeta(P) = 0\}} \text{index}(P) = +1$$

Among the consequences of this result is that the number of equilibria has to be odd. In particular it can never be zero and this gives a new proof of existence of equilibria.

In the search for global uniqueness of equilibria, there is another remark on the number of equilibria that is implicit in Dierker's result. He notices the following:

Theorem 11. (*Dierker, 1972*) *If the Jacobian of the excess supply function is positive at all Walras equilibria, there is exactly one equilibrium.*

The consequence of this result is that if there is only one equilibrium and the economy is regular, the index theorem also implies that the Jacobian evaluated at the equilibrium must be positive. So that generically, the condition is necessary and sufficient.

Giving an interpretation to theorem 11 is difficult. However, we quote an interpretation given by Varian (1975):

- By the use of Slutsky's equation, we can write the Jacobian matrix as a sum of substitution matrices, which are known to be positive definite, and income terms. Hence if at all equilibria income effects are *small* there will be only one equilibrium;
- Even if income effects are large, there will still exist a unique equilibrium if the marginal propensity to consume each good does not vary across the agents or if the *covariance* between net purchases of goods and the marginal propensity to consume each good is small;
- Suppose that the Jacobian matrix displays a *diagonal dominance* in that off diagonal terms are *small*, the theorem implies that if there are even number of Giffen goods at all equilibria, the equilibrium is unique.

2.5 Extending to infinite dimensions

Many economic problems have a commodity space that is infinite dimensional. We would like to extend the results of this chapter to this setting. It is not a trivial task for several reasons.

The first deals with the idea of positive cones in Banach spaces. In \mathbb{R}^n a positive cone is with respect to a basis. In general Banach spaces there is no natural way of defining “positivity” although of course for some, such as l_2 , there is a natural way of doing so. However, even if we can define a positive cone, it may have an empty interior. This means that we cannot use any results from differential analysis or differential topology that require open neighborhoods.

To fix ideas, let us give an example. The space l_2 is the set consisting of infinite sequences of real numbers

$$x = (x_1, x_2, \dots)$$

such that $\|x\|_2 = (|x_1|^2 + |x_2|^2 + \dots)^{1/2}$ is convergent. There is a natural positive cone l_2^+ given by sequences where each entry is a positive number. However, l_2^+ has an empty interior since for any sequence $y = (y_1, y_2, \dots) \in l_2^+$ and any $\epsilon > 0$, one can always find $y' \in l_2$ with at least one negative entry but such that $\|y - y'\| < \epsilon$.

Given this, we may be tempted to restrict ourselves only to those Banach spaces where the interior of the positive cone is nonempty, or we can take open neighborhoods even if we reach outside the positive cone. The problem is that we

are allowing agents to consume negative amounts of goods. This approach was exploited for instance by Kehoe et al (1989) by allowing negative consumption.

2.5.1 The setting

Another problem is that, even if the emptiness of the positive cone was discarded, we still don't know when we can make sure that individual demand functions are sufficiently differentiable or even continuous. As we will see in this chapter, this cannot always be guaranteed. We follow closely the notation of Araujo (1988).

Consumption and price spaces

Let X be a Banach space. We will refer to it as the consumption space. If there is a clear meaning, X^{++} (resp. X^+) will denote the positive (resp. non-negative) cone of X . The price space is X^* , the topological dual of X . Notice that in this chapter, and only in this chapter, the price space is not normalised.

Utility functions

Let A be a closed, convex, subset of X . Let $u : A \rightarrow \mathbb{R}$ be a continuous, strictly quasi-concave function. We will call it a utility function.

Recall that $u : A \rightarrow \mathbb{R}$ is called quasi-concave if

$$u(\alpha x + (1 - \alpha)y) \geq \min\{u(x), u(y)\}$$

for all $x, y \in A$ and all $\alpha \in [0, 1]$.

Demand functions

We say that a pair $(P, w) \in X^* \times \mathbb{R}$ is non-empty whenever the set $\{x \in A : P \cdot x = w\}$ is non-empty. For such a non-empty pair, the demand function $f(P, w)$ will be defined, if it exists, as the solution of

$$\sup_{\{x \in A : P \cdot x = w\}} u(x) \tag{2.2}$$

In this case we write $\exists f(P, w)$.

Contours

We will write C_α to denote $\{x \in A : u(x) \geq \alpha\}$. If $C_\alpha \neq \emptyset$, we will also write

$$w_P = \inf_{y \in C_\alpha} P \cdot y$$

2.5.2 General Banach spaces: Araujo's results

We would like to understand when demand functions are defined and, if they are so, when will they be continuously differentiable. Unfortunately, when X is an arbitrary Banach space, non-existence and non-differentiability seems to be the rule. The work of Araujo (1988) provides us with the definite results. In very loose words, for individual demand functions to be both well-defined and differentiable enough, we need certain concavity conditions on utility functions that can only be admitted when X is a Hilbert space.

A sufficient concavity condition

Definition 12. (Araujo's condition) We say that a utility function u satisfies Araujo's condition at the point \bar{x} , if there exists $\epsilon > 0$ such that for every h in $X_0 = \{h \in X : Du(\bar{x}) \cdot h = 0\}$

$$hD^2u(\bar{x})h \leq -\epsilon\|h\|^2$$

An interpretation of this condition is the following (cf. Shannon and Zame, 2002). If $u : X \rightarrow \mathbb{R}$ is twice continuously Gateaux differentiable and differentially strictly concave on a convex set $A \subset X_+$, then there is a constant $\epsilon > 0$ such that $hD^2u(\bar{x})h \leq -\epsilon\|h\|^2$ for all $h \in X$ and $\bar{x} \in A$. From Taylor's theorem, we get that u is *quadratically concave on A* , that is, that there is a constant $K > 0$ such that for each $x, y \in A$

$$u(y) \leq u(x) + Du(x) \cdot (y - x) - K\|y - x\|^2$$

In other words, Araujo's condition is a requirement that near any feasible bundle, utility differs from the linear approximation by an element that is at least quadratic in the distance to the given bundle. In other words, this *quadratic concavity* provides a measure of the extent to which distinct commodities are not perfect substitutes.

The general non-differentiability of demand

Araujo shows that if the commodity space X is not a Hilbert space, then one cannot find a single C^2 , strictly quasi-concave, utility function assuming its max-

imum on the interior of the consumption set and giving rise to a continuously differentiable demand function defined on its dual.

Theorem 13. *(Araujo, 1988) Suppose u is C^2 , $\exists f(\bar{P}, \bar{w}) \in \text{int}A$, $Du(f(\bar{P}, \bar{w})) \neq 0$ and Araujo's condition holds at \bar{x} . Then the demand is also C^1 at (\bar{P}, \bar{w}) . Moreover, any Banach space that has a C^2 utility satisfying Araujo's condition at a point \bar{x} is isomorphic to a Hilbert space.*

Theorem 14. *(Araujo, 1988) Suppose u is C^2 ; for (P, w) in a neighbourhood D of (\bar{P}, \bar{w}) such that $(P, w) \in D$ only for $w \neq 0$, $\exists f(P, w) \in \text{int}A$, $f(P, w)$ is C^1 and $Du(f(p, w)) \neq 0$. Then Araujo's condition holds for $\bar{x} = f(\bar{P}, \bar{w})$ and in particular X is isomorphic to a Hilbert space.*

The general non-existence of demand

Araujo shows that if the commodity space X is non-reflexive and if the consumption space has a non-empty interior then we cannot find a quasi-concave continuous utility function giving rise to a well defined demand function on it.

Theorem 15. *(Araujo, 1988) Suppose A is bounded and weakly closed, X is reflexive and u is weakly continuous. Then $\exists f(P, w)$ for every (P, w) non-empty.*

Theorem 16. *(Araujo, 1988) Suppose A is bounded, u is uniformly continuous and $\alpha \in \mathbb{R}$ is such that $\text{int}C_\alpha \neq \emptyset$ and that whenever $P \in X^*$, $\|P\| = 1$ then either there exists $x \in A$ with $P \cdot x < w_P$ and $\exists f(P, w_P)$ or $\inf_{x \in C_\alpha} P \cdot x$ is attained. Then X is reflexive.*

2.5.3 Separable utilities

After all the negative results reviewed in this chapter, we will continue in the next one by restricting the type of utility functions, after which we will after all be able to extend some of the ideas of the theory of general equilibrium to infinite dimensions. The objective will be to restrict ourselves only to “separable” utility functions; this changes our infinite-dimensional problem into a sequence of finite-dimensional problems.

Chapter 3

Determinacy of equilibria

3.1 Introduction

This chapter deals with generic determinacy of equilibria for infinite dimensional consumption spaces. This work could be seen as an infinite-dimensional analogue of Dierker and Dierker (1972), by characterising equilibria of an economy as a zero of the aggregate excess demand, and studying its transversality. In this case, we can use extensions of the transversality density theorem. Assuming separable utilities, we give a new proof of generic determinacy of equilibria. We define regular price systems in this setting and show that an economy is regular if and only if its associated excess demand function only has regular equilibrium prices. We also define the infinite equilibrium manifold and show that it has the structure of a Banach manifold.

3.2 The Fredholm approach

Amongst the most important results when modeling competitive markets is that of generic determinacy of equilibria. If we consider economies with a finite number of agents and a finite number of commodities, it is well known that almost all economies (i.e., almost all initial endowments) give rise to a finite number of competitive equilibria.

However, when an economy has an infinite number of commodities, determining whether equilibria is locally unique has presented us with many challenges. Araujo (1988), loosely speaking, shows that when the commodity set is a general Banach space a demand function will exist if and only if the commodity space is reflexive. He also shows that even if the demand function exists, it will be C^1 if and only if the commodity space is actually a Hilbert space.

Different approaches exist then to attack this problem. Because of Araujo's results, Kehoe et al (1989) study determinacy of equilibria where the commodity set is a Hilbert space. The disadvantage of this approach, as they put it, is that the price domain (and, implicitly, the consumption set) has an empty interior. This means that they are allowing, to some extent, negative prices and consumption.

A second approach consists in using a weakened version of differentiability. Shannon (1999) and Shannon and Zame (2002) introduce the notion of quadratic concavity and demonstrate that Lipschitz continuity of the excess spending map

is sufficient to yield generic determinacy. Because the nature of regularity for Lipschitz functions is weaker than for smooth functions, the set of regular economies is not open and dense nor is it the intersection of a countable family of open sets. Instead they use a measure-theoretic analogue of full Lebesgue measure for infinite dimensional spaces.

A third approach is to assume separable utilities (which we will precisely define later). In this case equilibrium conditions are described by Fredholm maps which are a good setting for transversality. Strictly speaking, the price space is the natural positive cone of the dual space of the commodity set (cf. Bewley 1972 and Prescott and Lucas 1972). However, with separable utility functions, only a small subset of the price space can support equilibria and there is no loss of information from discarding those elements that do not support equilibria. Pursuing this old idea we first review a result of Chichilnisky and Zhou (1998) who show that with separable utilities, individual demand functions are Fredholm maps and then they show that smooth infinite economies with separable utilities have locally unique equilibria. These results do not contradict Araujo's result seen in the previous chapter since we are restricting the price space to only a subset that admits equilibria.

In this chapter we will follow the approach of Chichilnisky and Zhou and extend these results. Our work could be seen as an infinite-dimensional analogue of Dierker and Dierker (1972), by characterising equilibria as a zero of aggregate excess demand. In this case, we can use extensions of the implicit function theorem and the transversality theorem. In section 3.2.5 we show that aggregate

excess demand functions also are Fredholm maps and compute their index. In section 3.3.2 we define the equilibrium set and show that it has the structure of a Banach manifold. We also define the notion of a regular equilibrium price system and show that an economy is regular if and only if all equilibrium prices of its associated excess demand function are regular. Finally, we give a new proof that regular economies are generic.

3.2.1 The Market

Commodity set

In order to use differential techniques, we assume that the consumption space is a separable topological vector space for which the interior of its positive cone is non-empty. We assume that the commodity space is a subset of $C(M, \mathbb{R}^n)$, where M is a compact ¹ manifold equipped with a smooth volume form dt such that

$$\int_M dt = 1$$

Example 1: In growth models a utility function on $C(M, \mathbb{R}^n)$ is a continuous-time version of a discounted sum of time-dependent utilities. Here M represents time.

Example 2: In finance, when the underlying parameters follow a diffusion process, a utility function on $C(M, \mathbb{R}^n)$ is the expectation of state-dependent utilities where M is the state space.

¹In principle, we should be able to allow M to be a non-compact manifold as long as we restrict ourselves to bounded continuous functions. However, this is an extension not followed in this thesis.

For more general consumption spaces we refer to Chichilnisky and Zhou (1995). We equip $C(M, \mathbb{R}^n)$ with the supremum norm

$$\|f\| = \sup_{t \in M} |f(t)|$$

The supremum norm is for instance useful in the neoclassical theory of growth where, under commonly made assumptions in the one sector growth model, capital/labor ratios are uniformly bounded over time.

Consumption space

The set $C(M, \mathbb{R}^n)$ has a natural positive cone $C^{++}(M, \mathbb{R}^n)$ given by the continuous functions $f : M \rightarrow \mathbb{R}^n$ such that the image is an n -vector with all entries positive. It can also be endowed with an inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \int_M f(t) \cdot g(t) dt$$

where $f(t) \cdot g(t)$ is the interior product in \mathbb{R}^n .

The **consumption space** is then $X = C^{++}(M, \mathbb{R}^n)$, the positive cone of $C(M, \mathbb{R}^n)$.

Exchange economies

We consider a finite number I of agents. An **exchange economy** is parametrized for each agent $i = 1, \dots, I$ by their initial endowments $\omega_i \in X$ and their individual demand functions $f_i : S \times (0, \infty) \rightarrow X$.

Utility functions

The maps $f_i(P(t), w)$ are solutions to the optimization problem

$$\max_{\langle P(t), y \rangle = w} W_i(y)$$

where $W_i(x)$ is a separable utility function, i.e., it can be written as

$$W_i(x) = \int_M u^i(x(t), t) dt$$

We assume, as in section 2.2.1, that $u^i : \mathbb{R}_{++}^n \times M \rightarrow \mathbb{R}$ is a strictly monotonic, concave, C^2 function where $\{y \in \mathbb{R}_{++}^n : u^i(y, t) \geq u^i(x, t)\}$ is closed. We have the following theorem.

Theorem 17. *(Chichilnisky and Zhou, 1998) Under the above assumptions on $u^i : \mathbb{R}_{++}^n \times M \rightarrow \mathbb{R}$, the function*

$$W_i(x) = \int_M u^i(x(t), t) dt$$

is strictly monotonic, concave over X and twice Frechet differentiable.

Prices

Strictly speaking, the price space is the natural positive cone of the dual space of the commodity set (cf. Bewley 1972 and Prescott and Lucas 1972). In our case, the price space is the positive cone of the dual space of $C(M, \mathbb{R}^n)$. However, with separable utility functions, only a small subset of the price space can support equilibria and there is no loss of information from discarding those elements that

do not support equilibria. We can just restrict it to a subset that can support equilibria and by discarding the rest we do not lose any information.

Definition 18. A **price** $P : C(M, \mathbb{R}^n) \rightarrow \mathbb{R}$ is a bounded and linear real-valued function on $C(M, \mathbb{R}^n)$ which gives non-negative values to any element of $C^{++}(M, \mathbb{R}^n)$.

Theorem 19 below shows that with separable utilities, prices will actually be in the consumption space X . This simplifies calculations since we will be able to pair a price P and a consumption or endowment by taking the inner product in $C(M, \mathbb{R}^n)$.

Theorem 19. (*Chichilnisky and Zhou, 1998*) *If $\omega_i \in X$ for all i , then under the above assumptions, every equilibrium price P belongs to $C^{++}(M, \mathbb{R}^n)$.*

The idea behind the proof of Theorem 19 is that a competitive equilibrium is Pareto efficient by the first welfare theorem and so it maximizes a weighted sum of individual utilities. So consider

$$\bar{u}(y, t) = \max_{x_i \in C^{++}(M, \mathbb{R}^n)} \sum_{i=1}^I \frac{1}{\lambda_i} u_i(x, t)$$

subject to $\sum_{i=1}^I x_i \leq y$. Finally, notice that $\bar{u}'(\omega, t) = \lambda P$ for some $\lambda > 0$. Therefore, $\omega_i \in C^{++}(M, \mathbb{R}^n)$ implies that $P \in C^{++}(M, \mathbb{R}^n)$

As in finite dimensions, we will normalize prices, and define:

Definition 20. The **price space** is $S = \{P \in C^{++}(M, \mathbb{R}^n) : \|P\| = 1\}$

3.2.2 Individual Demand Functions

Chichilnisky and Zhou (1998) also show that for separable utilities the individual demand functions $f_i : S \times (0, \infty) \rightarrow X$ satisfy the following properties below. The reader may wish to consult appendix A.2.2 for the definitions of Fredholm maps.

1. $\langle P(t), f_i(P(t), w) \rangle = w$ for any $P \in S$ and for any $w \in (0, \infty)$;
2. Let u_x denotes the partial derivative of u with respect to x . Then,

$$u_x^i(f_i(P(t), w), t) = \lambda P(t)$$

for some $\lambda > 0$ and for all $P \in S$;

3. $f_i : S \times (0, \infty) \rightarrow X$ is a diffeomorphism;
4. $f_i : S \times (0, \infty) \rightarrow X$ is a Fredholm map of index zero.

3.2.3 Extension to Infinite Dimensions of Debreu's determinacy

We finally will mention the main theorem of Chichilnisky and Zhou. Their result is an infinite-dimensional analogue of Debreu's seminal paper (1970). The two proofs are very similar, by exploiting arguments of Sard's theorem and Smale's extension to infinite dimensions.

For an economy $\omega = (\omega_1, \dots, \omega_I) \in X^I$ let $E(\omega)$ denote the set of $P \in S$ satisfying the following equality

$$\sum_{i=1}^I f_i(P(t), \langle P(t), \omega_i \rangle) = \sum_{i=1}^I \omega_i$$

We have the following result.

Theorem 21. (*Chichilnisky and Zhou, 1998*) *There is a dense subset V of X^I , the space of endowments, such that $E(\omega)$ is discrete for any $\omega \in V$, and for each such discrete point ω , locally the equilibrium in $E(\omega)$ depends continuously on ω .*

3.2.4 Aggregate Excess Demand Functions

In this chapter we assume that the utility functions u^i are fixed, so that the only parameters defining an economy are the initial endowments. Denote $\omega = (\omega_1, \dots, \omega_I) \in \Omega = X^I$. For a fixed economy $\omega \in \Omega$ the **aggregate excess demand function** is a map $Z_\omega : S \rightarrow C(M, \mathbb{R}^n)$ defined by

$$Z_\omega(P) = \sum_{i=1}^I (f_i(P, \langle P, \omega_i \rangle) - \omega_i)$$

We also define $Z : \Omega \times S \rightarrow C(M, \mathbb{R}^n)$ by the evaluation

$$Z(\omega, P) = Z_\omega(P)$$

Definition 22. We say that $P \in S$ is an **equilibrium** of the economy $\omega \in \Omega$ if $Z_\omega(P) = 0$. We denote the **equilibrium set**

$$\Gamma = \{(\omega, P) \in \Omega \times S : Z(\omega, P) = 0\}$$

We wish to explore the structure of aggregate excess demand functions. We

first show the well-known result that the excess demand defines a vector field on the price space.

Proposition 1. The excess demand function $Z_\omega : S \rightarrow C(M, \mathbb{R}^n)$ of economy $\omega \in \Omega$ is a vector field on S .

Proof. Since $\langle P, f_i(P, y) \rangle = y$ for any $P \in S$ and for any $y \in (0, \infty)$, then

$$\begin{aligned} \langle P, Z_\omega(P) \rangle &= \left\langle P, \sum_{i=1}^I (f_i(P, \langle P, \omega_i \rangle) - \omega_i) \right\rangle \\ &= \sum_{i=1}^I \langle P, f_i(P, \langle P, \omega_i \rangle) \rangle - \sum_{i=1}^I \langle P, \omega_i \rangle \\ &= \sum_{i=1}^I \langle P, \omega_i \rangle - \sum_{i=1}^I \langle P, \omega_i \rangle \\ &= 0 \end{aligned}$$

□

Denote by TS the tangent bundle of S and TS_0 its zero section. We can then interpret Z_ω as a section of TS and an equilibrium as a point where this section intersects TS_0 .

3.2.5 The Fredholm Index of the Excess Demand

In order to use techniques of differential topology in infinite dimensions, we require our maps to be Fredholm. We now show that this is the case for the excess demand function. Again, the reader may wish to consult appendix A.2.2 for the definitions of Fredholm maps.

Theorem 23. *The excess demand function $Z_\omega : S \rightarrow C(M, \mathbb{R}^n)$ of economy $\omega \in \Omega$ is a Fredholm map of index zero.*

The proof of Theorem 23 is rather computational, but it consists of two parts. The first is to show that Df_i can be written as the sum of an invertible operator plus a finite rank operator and hence it is a Fredholm map of index zero. The second part consists of explicitly writing DZ_ω in terms of the Df_i 's and once again showing that it can be written as the sum of an invertible operator plus a finite rank operator.

Before moving into the proof, we wish to remind the reader of Theorem 19. That is, that even though we define a *price* to be an element of the positive cone of the **dual** space of $C(M, \mathbb{R}^n)$, by Theorem 19, a price can be identified with an element of the positive cone of $C(M, \mathbb{R}^n)$ rather than its dual, i.e., a price is a bounded map from M to \mathbb{R}_{++}^n .

Proof. (of Theorem 23)

Recall that the consumers' problem is given by

$$\max_{x \in X} W_i(x) \quad \text{s.t.} \quad \langle P, x \rangle = w$$

where

- $X = C^{++}(M, \mathbb{R}^n)$;
- $W_i : X \rightarrow \mathbb{R}$ is given by $W_i(x) = \int_M u^i(x(t), t) dt$;
- $u^i : \mathbb{R}_{++}^n \times M \rightarrow \mathbb{R}$;

- In principle, P is an element of the positive cone of the dual of $C(M, \mathbb{R}^n)$.

However, we have shown in the thesis that with separable utilities, actually

$$P \in C^{++}(M, \mathbb{R}^n);$$

- Furthermore, we normalise so that $P \in S = \{P \in C^{++}(M, \mathbb{R}^n) : \|P\| = 1\}$;
- $w \in (0, \infty)$.

Notice that $P \in S$ and $w \in (0, \infty)$ are independent (i.e., exogenously determined) variables of the problem.

Now, because of the assumptions that we have placed on the utility functions u^i (smoothness, concavity, monotonicity), this implies that for each $P \in S$ and for each $w \in (0, \infty)$ the optimization problem has a unique solution that we will denote by $f_i(P, w)$ where $f_i : S \times (0, \infty) \rightarrow X$.

The first order optimality conditions can then be written as:

$$w = \langle P, f_i(P, w) \rangle \tag{3.1}$$

$$DW_i(f_i(P, w)) = \lambda_i(P, w) \cdot P \tag{3.2}$$

where DW_i denotes the Fréchet derivative of $W_i : X \rightarrow \mathbb{R}$ and $\lambda_i : S \times (0, \infty) \rightarrow \mathbb{R}$ is a Lagrange multiplier.

The strategy is to calculate the total derivatives of equations (3.1) and (3.2) and solve for $Df_i(P, w)$. We will exploit the simplicity of $W_i(x)$ written in terms

of u^i . Hence, we first write equations (3.1) and (3.2) as

$$w = \langle P, f_i(P, w) \rangle \quad (3.3)$$

$$u_x^i(f_i(P, w), t) = \lambda_i(P, w) \cdot P \quad (3.4)$$

Taking total derivatives on both sides of equations (3.3) and (3.4) we get

$$Dw = f_i(P, w) + \langle P, Df_i(P, w) \rangle$$

$$u_{xx}^i(f_i(P, w), t) \cdot Df_i(P, w) = \lambda_i(P, w) + P \cdot D\lambda_i(P, w)$$

where we write $\langle P, Df_i(P, w) \rangle$ to denote the linear transformation Df_i composed with the linear transformation P .

Simplifying, and remembering that since $u^i(x)$ is concave, the linear transformation (u_{xx}^i) is negative definite and hence (u_{xx}^i) is invertible for each t , we now have

$$Dw = f_i(P, w) + \langle P, Df_i(P, w) \rangle \quad (3.5)$$

$$Df_i(P, w) = \lambda_i(P, w) (u_{xx}^i)^{-1} + (u_{xx}^i)^{-1} P \cdot D\lambda_i(P, w) \quad (3.6)$$

Making a substitution of the expression of Df_i found in (3.6) into Dw of equation (3.5), and remembering that P is linear, we get

$$\begin{aligned}
Dw &= f_i(P, w) + \langle P, Df_i(P, w) \rangle \\
&= f_i(P, w) + \langle P, \lambda_i(P, w) (u_{xx}^i)^{-1} + D\lambda_i(P, w) (u_{xx}^i)^{-1} P \rangle \\
&= f_i(P, w) + \langle P, \lambda_i(P, w) (u_{xx}^i)^{-1} \rangle + \langle P, D\lambda_i(P, w) (u_{xx}^i)^{-1} P \rangle \\
&= f_i(P, w) + \lambda_i(P, w) (u_{xx}^i)^{-1} P + D\lambda_i(P, w) \langle P, (u_{xx}^i)^{-1} P \rangle
\end{aligned}$$

Therefore,

$$D\lambda_i(P, w) = \frac{1}{\langle P, (u_{xx}^i)^{-1} P \rangle} [Dw - f_i(P, w) - \lambda_i(P, w) (u_{xx}^i)^{-1} P] \quad (3.7)$$

where the denominator $\langle P, (u_{xx}^i)^{-1} P \rangle$ does not vanish since P and $(u_{xx}^i)^{-1}$ are positive operators.

We substitute the expression of $D\lambda_i$ found in (3.7) into (3.6) to get,

$$\begin{aligned}
Df_i(P, w) &= \lambda_i(P, w) (u_{xx}^i)^{-1} + D\lambda_i(P, w) (u_{xx}^i)^{-1} P \\
&= \lambda_i(P, w) (u_{xx}^i)^{-1} + \\
&\quad + \frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} [Dw - f_i(P, w) - \lambda_i(P, w) (u_{xx}^i)^{-1} P]
\end{aligned}$$

What we have shown is that $Df_i(P, w)$ can be written as the sum of the invertible operator

$$\lambda_i(P, w) (u_{xx}^i)^{-1} + \frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} Dw$$

and the finite rank operator

$$-\frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} [f_i(P, w) + \lambda_i(P, w)(u_{xx}^i)^{-1} P]$$

Now, let $w = \langle P, \omega_i \rangle$ and recall that $Z_\omega : S \rightarrow C(M, \mathbb{R}^n)$ is given by

$$Z_\omega(P) = \sum_{i=1}^I (f_i(P, \langle P, \omega_i \rangle) - \omega_i)$$

and so $DZ_\omega : TS \rightarrow TC(M, \mathbb{R}^n)$ is given by

$$\begin{aligned} DZ_\omega(P) &= \sum_{i=1}^I Df_i(P, w) \\ &= \sum_{i=1}^I \left\{ \lambda_i(P, w) (u_{xx}^i)^{-1} + \frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} Dw \right\} + \\ &\quad + \sum_{i=1}^I \left\{ -\frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} [f_i(P, w) + \lambda_i(P, w)(u_{xx}^i)^{-1} P] \right\} \end{aligned}$$

Finally, noticing again that since $u^i(x)$ is concave, the linear transformation (u_{xx}^i) is negative definite and hence (u_{xx}^i) is invertible. Additionally, the sum of negative-definite linear transformations is again negative-definite. Hence,

$$\sum_{i=1}^I \left\{ -\frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} [f_i(P, w) + \lambda_i(P, w)(u_{xx}^i)^{-1} P] \right\}$$

has finite rank, and

$$\sum_{i=1}^I \left\{ \lambda_i(P, w) (u_{xx}^i)^{-1} + \frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} Dw \right\}$$

is invertible. Therefore, DZ_ω is written as the sum of an invertible operator and an operator of finite rank which in turn implies that Z_ω is a Fredholm map of index zero.

□

3.3 Determinacy of equilibria

In this section we wish to show parametric transversal density. We first need to give a manifold structure to the equilibrium set Γ . Our result is a straightforward extension of Balasko's work (1975b, 1988).

3.3.1 Regular Values

Proposition 2. The derivative of the map $Z : \Omega \times S \rightarrow TS$ is a surjective map. In particular, it has 0 as a regular value.

Proof. Notice that because of the properties of individual demand functions 3.2.2 we get that Z is differentiable. We need to compute the derivative

$$DZ : T(\Omega \times S) \rightarrow T(TS)$$

Linearizing $Z(\omega, P)$ to first order in ϵ , dropping the $O(\epsilon^2)$ terms, and letting

$y_i = \langle P, \omega_i \rangle$, we get

$$\begin{aligned}
& Z(\omega_1 + \epsilon k_1, \dots, \omega_I + \epsilon k_I, P + \epsilon h) \\
&= \sum f_i(P + \epsilon h, \langle P + \epsilon h, \omega_i + \epsilon k_i \rangle) - \sum (\omega_i + \epsilon k_i) \\
&= \sum f_i(P + \epsilon h, \langle P, \omega_i \rangle + \epsilon \langle P, k_i \rangle + \epsilon \langle h, \omega_i \rangle) - \sum \omega_i - \epsilon \sum k_i \\
&= \sum [f_i(P, \langle P, \omega_i \rangle) + \epsilon (D_{y_i} f_i)_{(P, \langle P, \omega_i \rangle)}(\langle P, k_i \rangle) + \\
&\quad + \epsilon (D_{y_i} f_i)_{(P, \langle P, \omega_i \rangle)}(\langle h, \omega_i \rangle) + \epsilon (D_P f_i)_{(P, \langle P, \omega_i \rangle)}(h)] - \sum \omega_i - \epsilon \sum k_i \\
&= Z(\omega_1, \dots, \omega_I, P) + \\
&\quad + \epsilon \sum [(D_P f_i)_{(P, \langle P, \omega_i \rangle)}(h) + (D_{y_i} f_i)_{(P, \langle P, \omega_i \rangle)}(\langle P, k_i \rangle + \langle h, \omega_i \rangle) - k_i]
\end{aligned}$$

Or in matrix form, $DZ_{(\omega, P)} =$

$$\left(\begin{array}{c|c} \overbrace{0 \dots 0}^{i=1, \dots, I} & 1 \\ \hline \underbrace{(D_{y_i} f_i)_{(P, \langle P, \omega_i \rangle)}(\langle P, - \rangle) - Id}_{i=1, \dots, I} & \sum_i (D_P f_i)_{(P, \langle P, \omega_i \rangle)}(-) + \\ & + \sum_i (D_{y_i} f_i)_{(P, \langle P, \omega_i \rangle)}(\langle -, \omega_i \rangle) \end{array} \right)$$

where the *dashes* simply indicate that the left side of the matrix acts on (k_1, \dots, k_I) while the right acts on h .

To compute the cokernel let

$$DZ_{(\omega, P)}(k_1, \dots, k_I, h) = (Q, \dot{Q}) \in T(TS)$$

We need to solve for (k_1, \dots, k_I, h) . We first observe that $h = Q$. The second row would then be,

$$\sum \{[(D_{y_i} f_i)(\langle P, k_i \rangle) - (k_i)] + [(D_P f_i)(Q)] + [(D_{y_i} f_i)(\langle Q, \omega_i \rangle)]\} = \dot{Q}$$

Then

$$\sum [(D_{y_i} f_i)(\langle P, k_i \rangle) - (k_i)] = H(Q, \dot{Q}) \quad (3.8)$$

where

$$H(Q, \dot{Q}) = \dot{Q} - \sum \{[(D_P f_i)(Q)] + [(D_{y_i} f_i)(\langle Q, \omega_i \rangle)]\}$$

But for every $i = 1, \dots, I$, the map $k_i \mapsto (D_{y_i} f_i)(\langle P, k_i \rangle) - (k_i)$ is onto. And, therefore, so is DZ .

□

3.3.2 The Infinite Equilibrium Manifold

Knowing that 0 is a regular value of Z we would like to give the equilibrium set Γ the structure of a Banach manifold.

First let us define the notion of transversality.

Definition 24. (Abraham and Robbin, 1967, p.45) Let X and Y be C^1 manifolds, $f : X \rightarrow Y$ a C^1 map, and $W \subset Y$ a submanifold. We say that f is **transversal to W at a point $x \in X$** , in symbols $f \pitchfork_x W$, iff, where $y = f(x)$, either $y \notin W$ or $y \in W$ and

1. the inverse image $(T_x f)^{-1}(T_y W)$ splits, and
2. the image $(T_x f)(T_x X)$ contains a closed component to $T_y W$ in $T_y Y$

We say f is **transversal to W** , in symbols $f \pitchfork W$, iff $f \pitchfork_x W$ for every $x \in X$.

Quinn (1970) defines a C^∞ **representation of maps** $\rho : A : M \rightarrow N$ consisting of Banach manifold A, M, N together with a function $\rho : A \rightarrow C^\infty(M, N)$ such that the evaluation map

$$Ev_\rho : A \times M \rightarrow N, \quad (a, m) \mapsto \rho_a(m)$$

is C^∞ . In our situation, $Ev_\rho : A \times M \rightarrow N$ corresponds to $Z : \Omega \times S \rightarrow TS$.

Suppose we have a C^∞ map $F : W \rightarrow N$ which is transversal to Ev_ρ . If we form the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & W \\ h \downarrow & & F \downarrow \\ A \times M & \xrightarrow{Ev_\rho} & N \\ \pi_A \downarrow & & \\ A & & \end{array} \quad (3.9)$$

where $P = (Ev_\rho \times F)^{-1}(\Delta_N)$ and Δ_N denotes the diagonal in $N \times N$, then P is a C^∞ Banach manifold, and $\pi_A \circ h$ is a C^∞ map.

Theorem 25. *The equilibrium set Γ is a C^∞ Banach manifold of dimension equal to $\dim \ker DZ$. We shall call it the **equilibrium manifold**. Furthermore the natural projection map $pr_\Omega : \Omega \times S|_\Gamma \rightarrow \Omega$ is a C^∞ map.*

Proof. Notice that the inclusion $T_0S \rightarrow TS$ is a C^∞ map. We also know from Proposition 2 that DZ is surjective, so it has 0 as a regular value. Then, we can form the pullback diagram

$$\begin{array}{ccc}
 \Gamma & \longrightarrow & T_0S \\
 \downarrow & & \downarrow \\
 \Omega \times S & \xrightarrow{Z} & TS \\
 \text{\scriptsize pr_Ω} \downarrow & & \\
 \Omega & &
 \end{array}$$

and as in diagram (3.9) we get that Γ is a C^∞ Banach manifold and the natural projection map is a C^∞ map. \square

3.3.3 Regular Economies

Definition 26. We say that an economy is **regular** (resp. **critical**) if and only if ω is a regular (resp. critical) value of the projection $pr : \Omega \times S \rightarrow \Omega$ restricted to $\Gamma \subset \Omega \times S$.

Definition 27. Let Z_ω be the excess demand of economy ω . A price system $P \in S$ is a **regular equilibrium price system** if and only if $Z_\omega(P) = 0$ and $DZ_\omega(P)$ is surjective.

We would like to compare the set of regular economies with those economies whose excess demand function has only regular prices. In finite dimensions these two sets are equal (Dierker, 1982).

Quinn (1970) will tell us that these two sets coincide; precisely, in diagram (3.9), $\rho_a \pitchfork F$ if and only if a is a regular value of $\pi_A \circ h$. And so we get,

Proposition 3. The economy $\omega \in \Omega$ is regular if and only if all equilibrium prices of Z_ω are regular.

Proof. Consider the diagram

$$\begin{array}{ccc}
 \Gamma & \longrightarrow & T_0 S \\
 \downarrow & & \downarrow \\
 \Omega \times S & \xrightarrow{z} & TS \\
 \text{\textit{pr}}_\Omega \downarrow & & \\
 \Omega & &
 \end{array}$$

Quinn's result says that the excess demand Z_ω is transversal to $T_0 S$ if and only if ω is a regular value of $\text{\textit{pr}}_\Omega$. \square

3.3.4 Determinacy of equilibria

We would now like to understand how big is the set of economies that give an excess demand function with all equilibrium prices being regular. For that, we need a couple of definitions.

A **left Fredholm map** is a map of Banach manifolds of class at least C^1 whose derivative at each point has closed image and finite dimensional kernel.

Definition 28. (Quinn, 1970) A map $f : X \rightarrow Y$ between topological spaces is **σ -proper** if there is a covering $(X_n)_{n \in \mathbb{N}}$ of X such that $f|_{X_n} : X_n \rightarrow Y$ is a proper map.

Quinn has also proved that a transversal density theorem holds in infinite dimensions.

Theorem 29. (Quinn, 1970) Let $\rho : A : M \rightarrow N$ be a C^∞ representation of left Fredholm maps, M separable, and $F : W \rightarrow N$ a C^∞ σ -proper left Fredholm map. If further

1. F is transversal to Ev_ρ , and
2. each ρ_a satisfies that for each $m \in M$ and $w \in W$ such that $\rho_a(m) = F(w)$, then $(\text{im } T_m \rho_a) \cap (\text{im } T_w F)$ is finite dimensional

then the set of a with $\rho_a \pitchfork F$ is residual in A .

The infinite-dimensional transversal density theorem can be used to give us an alternative proof that almost all economies are regular.

Theorem 30. The set of regular economies is residual in Ω .

Proof. Observe that the inclusion $T_0S \rightarrow TS$ given by $P' \mapsto (P', 0)$ is σ -proper since its domain consists of one set restricted to which the inclusion is proper since the inclusion map is continuous. Now, $T_0S \rightarrow TS$ is also left Fredholm since the derivative of the inclusion map is again the inclusion map, so it is continuous (and so has a closed image) and has finite dimensional kernel.

We also know that $Z(\omega, P)$ has 0 as a regular value since $DZ(\omega, P)$ is surjective.

All that we need to show is that for each $P \in S$ and $P' \in T_0S$ such that $Z_\omega(P) = I(P')$, where $I : T_0S \rightarrow TS$ is the inclusion map, we have

$$(\text{im } T_P Z_\omega) \cap (\text{im } T_x I)$$

is finite dimensional. But this follows immediately if we notice that $Z_\omega(P) = I(P')$ whenever P is an equilibrium, i.e. a zero of the vector field Z_ω . In this case $(\text{im} T_P Z_\omega) = 0$ and $(\text{im} T_x I) = 0$. Therefore, Theorem 29 implies the result. \square



Chapter 4

The Number of Equilibria

4.1 Introduction

In the previous chapter we established the fact that generically equilibria will be determinate. Since we normalized the price space, there are actually only finitely many equilibria. Intuitively, this means that it makes sense to count equilibria and in this chapter we establish a formula that will enable us to do so. More importantly, it will give us conditions that guarantee global, and not just local, uniqueness.

Counting the number of equilibria is an area that still remains largely unexplored for infinite-dimensional consumption spaces; I believe I am the first author to explore this issue. We mentioned in section 2.4 that when the consumption space is finite dimensional, Dierker (1972) gave the first solution to this problem, and constructed an index theorem that showed that the number of equilibria is generically odd. He does this by interpreting the excess demand function as a

vector field on the space of prices, and noticing that equilibria are the zeros of this vector field. He defines the notion of index of an equilibrium price system and shows that the sum of these indices is constant and equal to 1.

In this chapter, it is our aim to construct an infinite dimensional analogue of Dierker's result: that the number of equilibria of smooth infinite economies is odd. We present an analytical notion that has not appeared in the economic literature which is that of a Z-Rothe vector field. When the aggregate excess demand function defines a Z-Rothe vector field, it allows us to construct an index theorem on the normalized infinite dimensional price space.

We start in section 4.2 by defining the notions of Fredholm group, Rothe set and Rothe group as established by Tromba (1978). These tools are useful in defining the index of zeros of vector fields on Banach manifolds.

In section 4.3 we study the degree of a Fredholm map. We define the useful concepts of Fredholm structure and orientation. Then we mention one of the pillars of topology in infinite dimensions: the Smale-Sard theorem. We finally review the notion of degree of a Rothe map. All of these concepts allow us to generalize the idea of a map that is orientation-preserving or orientation-reversing.

In section 4.4 we review the concept of vector fields on Banach manifolds and some additional structures on these vector fields. In particular, we mention Z-Palais-Smale, Z-Fredholm and Z-Rothe vector fields. Then in section 4.5 we study how Tromba defined the Euler characteristic of these vector fields, and

in section 4.6 we review why the Euler characteristic is invariant under homotopy.

We use all of these results in section 4.7 to show that the excess demand function is a non-degenerate Z-Rothe vector field and in section 4.8 we prove the index theorem for smooth infinite economies. We also provide some concluding remarks on global uniqueness of equilibria.

4.2 The Fredholm group and the Rothe set

In this section we review the results from Tromba (1978). The most important concepts are the notion of Fredholm group and the Rothe set. The ultimate goal is to understand the notion of “index” of a singularity of a vector field on an infinite-dimensional manifold.

We start in section 4.2.1 by reviewing the generalized Fredholm group. This is a Lie group of invertible maps which has two components. This is a model for eventually generalizing the idea of orientation-preserving and orientation-reversing maps.

There is however the complication that the Fredholm group is not an open set. Hence, we can consider a slightly larger set called the generalized Rothe group as studied in section 4.2.3.

4.2.1 The Fredholm group

Let E be a Banach space. Recall that $\mathcal{L}(E)$ is the set of linear continuous maps from E to itself and $C(E)$ is the linear space of compact linear maps from E to itself.¹

We start by defining the set of compact perturbations of the identity map.

Definition 31. Let

$$\mathcal{L}_C(E) = \{T : T = I + C, I \text{ the identity}, C \in C(E)\}$$

If $T \in \mathcal{L}_C(E)$ then T is Fredholm of index zero (see Tromba, 1978).

Denote by $GL(E)$ the general linear group of E ; that is, the set of invertible linear maps in $\mathcal{L}(E)$.

Kuiper (1965) shows that if E is a Hilbert space, then $GL(E)$ is contractible as a topological space (so it is topologically trivial). Tromba remarks that this implies that tangent bundles of infinite-dimensional Banach manifolds are trivial which means, in some sense that $GL(E)$ is too big a group for much finite-dimensional topology to go through in infinite dimensions. Tromba therefore introduces a (Lie) subgroup $GL_C(E)$ of $GL(E)$.

Definition 32. We define the generalized **Fredholm group** to be

$$GL_C(E) = \mathcal{L}_C(E) \cap GL(E)$$

¹Eventually E will become the tangent space at a point on the price space.

Tromba also shows that $GL_C(E)$ has two components in the following sense:

Proposition 4. (Tromba 1978) $GL_C(E)$ is a Lie subgroup of $GL(E)$ for which $\pi_0(GL(E)) = \mathbb{Z}_2$, the integers mod 2. Thus $GL_C(E)$ has two components, which remains true if E does not have a Schauder basis.

Clearly the identity is in $GL_C(E)$, and since $GL_C(E)$ has two components, denote by $GL_C^+(E)$ the component of the identity in $GL_C(E)$ and by $GL_C^-(E)$ the other component.

Tromba also provides us with an example of an element of $GL_C^-(E)$. Write $E = E_0 \times R$ where R is a one-dimensional subspace of E . Define $J \in GL_C(E)$ by

$$J(x, y) = (x, -y), \quad (x, y) \in E_0 \times R$$

Then J is an element of $GL_C^-(E)$.

4.2.2 The Rothe set

Tromba remarks that the Fredholm group $GL_C(E)$ is not open in $GL(E)$. So in order to study vector fields on Banach manifolds, he introduces a slightly larger set than $GL_C(E)$. We would like this new set, denoted by $GR(E)$, to be such that

1. $GR(E)$ also has two components
2. $GR(E)$ contains $GL_C(E)$ as a subset

3. and yet $G\mathcal{R}(E)$ is open in $G\mathcal{L}(E)$

The idea is to generalize the idea of compact perturbations of the identity to compact perturbations of a suitable larger class of maps.

First consider the definition of $\mathcal{S}(E)$.

Definition 33. Let $\mathcal{S}(E) \subset G\mathcal{L}(E)$ be the maximal starred neighborhood of the identity in $G\mathcal{L}(E)$. Formally,

$$\mathcal{S}(E) = \{T \in G\mathcal{L}(E) : (\alpha T + (1 - \alpha)I) \in G\mathcal{L}(E), \forall \alpha \in [0, 1]\}$$

Tromba shows that $\mathcal{S}(E)$ has the following properties:

Theorem 34. (*Tromba, 1978*)

- $\mathcal{S}(E)$ is open in $G\mathcal{L}(E)$
- $T \in \mathcal{S}(E)$ if and only if $T^{-1} \in \mathcal{S}(E)$
- $\mathcal{S}(E)$ is contractible to the identity

Now consider the following definition:

Definition 35. The **Rothe set** $\mathcal{R}(E)$ is defined to be

$$\mathcal{R}(E) = \{A : A = T + C, T \in \mathcal{S}(E), C \in C(E)\}$$

4.2.3 The Rothe group

Definition 36. The **generalized Rothe group** $GR(E)$ is defined to be

$$GR(E) = \mathcal{R}(E) \cap GL(E)$$

Some properties of $\mathcal{R}(E)$ and $GR(E)$:

Theorem 37. (Tromba, 1978) *The Rothe set $\mathcal{R}(E)$ and the generalized Rothe group $GR(E)$ satisfy the following:*

1. $\mathcal{L}_C(E) \subset \mathcal{R}(E)$;
2. $GL_C(E) \subset GR(E)$;
3. $\mathcal{R}(E)$ is open in $\mathcal{L}(E)$;
4. $GR(E)$ is open in $GL(E)$;
5. $GR(E)$ is homotopically equivalent to $GL_C(E)$;
6. $GR(E)$ has two components which we shall denote by $GR^+(E)$ and $GR^-(E)$.

4.3 Degree of a Fredholm map

Since we will be counting equilibrium prices, it is worth exploring the idea of degree of a Fredholm map. This theory has been studied extensively in Elworthy and Tromba (1967, 1968).

This section is structured as follows. In section 4.3.1 we review Fredholm structures. Intuitively, this means that the linearised transition functions between

one chart and another have to be invertible and a compact perturbation of the identity. In section 4.3.2 the notion of orientation is studied. Most results will rely heavily on the Smale-Sard theorem that we mention in 4.3.3.

4.3.1 Fredholm structure

Definition 38. A Φ -structure or **Fredholm structure** M_ϕ modeled on E on a manifold M consists of a maximal atlas $\{(\mathcal{U}_i, \phi_i)\}$ for M , $\phi_i : \mathcal{U}_i \rightarrow E$ such that when defined, the derivative, $D(\phi_i \circ \phi_j^{-1})(x) \in GL_C(E)$.

4.3.2 Orientation

Definition 39. The Fredholm structure M_ϕ is said to be **orientable** if there is a subatlas of $\{\mathcal{U}_i, \phi_i\}$ with $D(\phi_i \circ \phi_j^{-1})(x) \in GL_C^+(E)$. A maximal subatlas will be called an **orientation** of M .

If M is connected, any orientable Fredholm structure on M will admit exactly two orientations.

4.3.3 Smale-Sard theorem

One of the most important results in differential topology is Sard's theorem. An analogue for infinite dimensions was discovered by Smale by using Fredholm maps. Recall that

Definition 40. Let M and N be smooth Banach manifolds. A C^1 map $f : M \rightarrow N$ is **Fredholm** if $Df_x : T_x M \rightarrow T_{f(x)} N$ is linear Fredholm for each $x \in M$. By

the index of f we mean the index of Df_x . If M is connected this does not depend on x . If M is not connected, we shall assume the index to be the same for all components. A Fredholm map of index n will be called a Φ_n map.

The Smale-Sard theorem below states that “most” values are regular.

Theorem 41. (*Smale-Sard*) Suppose that $f : M \rightarrow N$ is a C^r σ -proper Fredholm map between Banach manifolds where $r > \max(\text{index } f, 0)$. Then the set \mathcal{C} of regular values of f is a Baire subset of N . If f is proper, \mathcal{C} is open and dense.

4.3.4 $\Phi(I)$ –maps

Suppose that M, N are manifolds with Φ -structures M_ϕ, N_ψ modeled on E . A C^r map $f : M \rightarrow N$ will be called a $\Phi(I)$ –map from M_ϕ to N_ψ if

$$D(\psi_i \circ f \circ \phi_j^{-1})(\phi_j(x)) \in \mathcal{L}_C(E)$$

for all $x \in M$ and charts ψ_i, ϕ_j of M_ϕ, N_ψ for which it is defined. A $\Phi(I)$ –map is necessarily a Φ_0 map.

That is, a $\Phi(I)$ –map is a map between manifolds that in coordinates is of the form identity plus compact, and therefore it is a Fredholm map of index zero.

4.3.5 Oriented degrees

If $r \geq 2$ and M_ϕ, N_ψ have orientations and if f is proper we may apply Smale’s theorem to obtain an oriented degree for f just as in the finite-dimensional case;

namely, take a regular value y of f in N and let $\deg f$ be the algebraic number of points in $f^{-1}(y)$

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sgn} Df_x$$

where $\operatorname{sgn} Df_x = \pm 1$ depending on whether $D(\psi_i \circ f \circ \phi_j^{-1})(\phi_j(x))$ lies in $GL_C^+(E)$ or $GL_C^-(E)$ for oriented charts ψ_i, ϕ_j at $f(x), x$. If $f^{-1}(y) = \emptyset$, $\deg f = 0$. This Brouwer degree gives an invariant of f under proper C^r homotopies through proper $\Phi(I)$ -maps of M_ϕ into N_ψ .

4.3.6 Degree of a Rothe map

There is another notion of degree when one has a closed domain B of a Banach space E (or a Banach manifold M) whose boundary will be denoted by ∂B , together with a point $y \in E$ and a proper C^r Φ_0 -map $f : (B, \partial B) \rightarrow (E, E - \{y\})$, $r \geq 2$.

As above, we can define an integer $\deg(f, \partial B, y)$ by looking at the inverse image of a regular value of f lying on the component of y in $E - f(\partial B)$. If f is an identity plus compact field and B is bounded in E , this degree is called the **Leray-Schauder degree**.

Definition 42. A C^2 function $f : B \rightarrow E$, B a domain in E is a **Rothe map** if for each $x \in B \subset E$ the Frechet derivative $Df(x) \in \mathcal{R}(E)$.

Rothe maps are Fredholm maps of index zero. Moreover we have:

Theorem 43. (Tromba 1978) *A Rothe map f induces in a natural way a unique oriented Φ -structure B_ϕ on B^0 , the interior of B , with respect to which f is a $\Phi(I)$ -map.*

Let $f : (B, \partial B) \rightarrow (E, E - \{y\})$ be a proper Rothe map. Since f is proper $f(\partial B)$ is closed (Tromba, 1978) and $y \in \mathcal{O}$, \mathcal{O} the open component of y in $E - f(\partial B)$. Let B_ϕ be the orientable C structure on B^0 given by theorem 43. \mathcal{O} as an open submanifold of E inherits a natural orientation. Let $M = f^{-1}(\mathcal{O})$ and let M_ϕ be the oriented Φ -structure on M induced by B_ϕ . Then $f : M \rightarrow \mathcal{O}$ is a proper $\phi(I)$ -map from M_ϕ to \mathcal{O}_ϕ and therefore has a Brouwer degree, $\deg f$ given by

$$\deg f = \sum_{x \in f^{-1}(y)} \text{sgn} Df_x$$

We shall denote this degree by

$$\deg(f, B, y)$$

which is the degree of a Rothe map.

4.3.7 Properties of the degree of a Rothe map

The degree $\deg(f, B, y)$ satisfies the following properties that can be found in Tromba (1978). Recall that $f : (B, \partial B) \rightarrow (E, E - \{y\})$ where y is a regular value.

- If $z \in \mathcal{O}$, \mathcal{O} the open component of y in $E - f(\partial B)$ then

$$\deg(f, B, z) = \deg(f, B, y)$$

- If $\deg(f, B, y) \neq 0$ then there exists an $x \in B^0$ with $f(x) = y$
- Additivity: if $\gamma^{-1}(y) = \cup_{i=1}^n C_i$ where the C_i are disjoint compact sets and if $C_i \subset \mathcal{U}_i^0$, $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$, $i \neq j$ then

$$\deg(f, B, y) = \sum_i \deg(f, \mathcal{U}_i, y)$$

- Invariance under homotopy: if $f_t : (B, \partial B) \rightarrow (E, -\{y\})$, $0 \leq t \leq 1$ is a homotopy of Rothe maps then

$$\deg(f_0, B, y) = \deg(f_1, B, y)$$

4.3.8 Oriented degree and the Rothe group

When y is a regular value, a very nice interpretation can be given of this degree, an interpretation which is already implicit in Tromba's results. If y is regular, let x_1, \dots, x_k be the finite number of points in B^0 in $f^{-1}(y)$. Then

$$\deg(f, B, y) = \sum_{x_i \in f^{-1}(y)} \operatorname{sgn} Df(x_i)$$

where

$$\operatorname{sgn} Df(x_i) = \begin{cases} +1, & \text{if } Df(x_i) \in GR^+(E) \\ -1, & \text{if } Df(x_i) \in GR^-(E) \end{cases}$$

4.4 Vector fields on Banach manifolds

We now review three classes of vector fields on Banach manifolds. There will be special structures required on its zero set.

Let $X : M \rightarrow TM$ be a C^r , $r \geq 1$ vector field on a C^{r+1} Banach manifold, modeled on a Banach space E .

4.4.1 Z-Palais-Smale (ZPS) vector fields

Definition 44. A C^1 vector field X on a Banach manifold M is **Z-Palais-Smale** or ZPS if whenever $X(p) = 0$ the Frechet derivative

$$DX(p) \in \mathcal{L}_C(T_p M)$$

4.4.2 Z-Fredholm vector fields

Definition 45. A C^1 vector field is **Z-Fredholm** of index n if whenever $X(p) = 0$,

$$DX(p) \in \Phi_n$$

that is, $DX(p)$ is linear Fredholm of index n .

4.4.3 Z-Rothe (ZR) vector fields

Definition 46. A C^1 vector field is **Z-Rothe** or ZR if whenever $X(p) = 0$,

$$DX(p) \in \mathcal{R}(T_p M)$$

4.5 The Euler characteristic of vector fields

Tromba defines the Euler characteristic of ZPS and ZR vector fields. We start with general vector fields and the next subsection incorporates the nondegenerate case.

4.5.1 ZPS and ZR vector fields

Let M be a C^{r+1} , $r \geq 2$ paracompact manifold (perhaps with boundary) modeled on a Banach space E which admits an equivalent C^1 norm. Thus M admits C^2 partitions of unity.

Let $X : M \rightarrow TM$ be a C^2 ZPS vector field with a finite number of isolated zeros p_1, \dots, p_k in the interior of M .

Let $\phi_i : \mathcal{U}_i \rightarrow \mathcal{O}$ with $\phi_i(p_i) = 0 \in E$. Then X^{ϕ_i} , the principal part of X in the coordinate system ϕ_i , is a map $X^{\phi_i} : \mathcal{O} \rightarrow E$ with derivative $DZ^{\phi_i}(0) \in GL_C(E)$. For a sufficiently small closed ball B_i about 0, $X^{\phi_i} : B_i \rightarrow E$ is a proper Rothe map (this follows from the fact that $\mathcal{R}(E)$ is open in $\mathcal{L}(E)$), and hence has a local degree $\deg(X^{\phi_i}, B_i, 0)$. Tromba (1978) shows, using the homotopy property

of degree, that this does not depend on the coordinate patch ϕ_i .

The Euler characteristic $\chi(X)$ is given by the formula

$$\chi(X) = \sum \deg(X^{\phi_i}, B_i, 0) \quad (4.1)$$

and if X has no zeros we set $\chi(X) = 0$.

If X is a Z-Rothe vector field with finitely many zeros then 4.1 also defines the Euler characteristic of X .

4.5.2 Nondegenerate ZPS and ZR vector fields

There is a simpler interpretation of the Euler characteristic of ZPS or ZR vector fields if its zeros are nondegenerate.

Definition 47. A vector field $X : M \rightarrow TM$ is a **proper vector field** if the set of zeros of X form a compact subset of M .

If the zeros of a proper vector field are isolated then there are only finitely many of them. If in addition the vector field is ZPS (or ZR) and the zeros are in the interior of M the Euler characteristic is defined.

Definition 48. A zero P of a vector field X is **nondegenerate** if $DX(P) : T_P M \rightarrow T_P M$ is a bounded linear isomorphism.

Tromba (1978) shows that nondegenerate zeros are isolated.

When a proper ZPS or ZR vector field has only nondegenerate zeros one can give a particularly simple interpretation of the Euler characteristic. Let p be a nondegenerate zero of X . Then for $E = T_p M$, $DX(p) \in GR(E)$. Define

$$\text{sgn}DX(P) = \begin{cases} +1, & \text{if } DX(P) \in GR^+(T_P M) \\ -1, & \text{if } DX(P) \in GR^-(T_P M) \end{cases}$$

The Euler characteristic is then give by the simpler formula

$$\chi(X) = \sum_{p \in \text{zeros}(X)} \text{sgn}DX(p) \quad (4.2)$$

4.6 Invariance of Euler characteristic under homotopy

This section studies how the Euler characteristic behaves under homotopy.

Let $\pi_0 : M \rightarrow [0, 1]$ be a C^3 smooth fiber bundle over the unit interval with $\pi_0^{-1}(t) = M_t$ a C^2 Banach manifold.

Definition 49. Two C^2 proper ZPS vector fields $X_0 : M_0 \rightarrow TM_0$ and $X_1 : M_1 \rightarrow TM_1$ are **equivalent** ($X_0 \sim X_1$) if there is a C^2 proper vector field $X : M \rightarrow TM$ such that

1. $X_t = X|_{M_t} : M_t \rightarrow TM_t$;
2. For each t , X_t is ZPS;

3. The zeros of X are in the interior of M .

Theorem 50. (*Tromba, 1978*) Suppose $X_0 \sim X_1$. Then $\chi(X_0)$ and $\chi(X_1)$ are both defined and equal.

4.7 Excess demand function is a nondegenerate Z-Rothe vector field

Knowing that the excess demand function is a vector field on the price space, and that is a Fredholm map for which we know its index, we would like to give it the structure of a Z-Rothe vector field. In section 4.8 it will become clear that we need a vector field that is outward pointing, so we insist $-Z_\omega$ to be Z-Rothe.

Proposition 5. The negative of the excess demand function, $Z_\omega : S \rightarrow TS$ is a Z-Rothe vector field.

Proof. From the proof of theorem 23, we know that $-Z_\omega$ can be written as the sum of a finite rank operator and an invertible operator. All we need to show then is that

$$\alpha \left[-\lambda_i(P, w) (u_{xx}^i)^{-1} - \frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} Dw \right] + (1 - \alpha)I$$

is invertible for all $\alpha \in [0, 1]$. But this sum is just a homotopy of positive-definite operators.

□

4.8 The Index Theorem of Smooth Infinite Economies

Knowing that most economies are regular we need to find a right way of counting the number of equilibria. With an excess demand function that is a Fredholm map, we may use tools of infinite-dimensional differential topology that resembles the finite dimensional case. This is why the proof of the index theorem in our setting is very similar to that of Dierker (1972).

Suppose that the excess demand satisfies the boundary assumption of Dierker (1972), namely that if $P_n \in S$ and $P_n \rightarrow P \in \partial S$, then

$$\|Z_\omega(P_n)\| \rightarrow \infty.$$

Suppose also that Z_ω is bounded below and that there are only finitely many zeros. The final ingredient before proving the index theorem is to check that the excess supply function is a vector field that is outward pointing along the boundary of S . To see this, consider a sequence of prices $P_n \rightarrow P \in \partial S$. Since we have assumed that Z_ω is bounded from below and that $\|Z_\omega(P_n)\| \rightarrow \infty$, then the limit of $[1/\|Z_\omega(P_n)\|] Z_\omega(P_n)$ must converge to a point $z \in C^{++}(M, \mathbb{R}^n)$. Hence, Z_ω is inward-pointing and therefore $-Z_\omega$ is outward-pointing along ∂S .

4.8.1 The Index Theorem of Smooth Infinite Economies

Theorem 51. *Suppose that an aggregate excess demand function Z is bounded from below and that it satisfies the boundary assumption. Suppose also that Z has only finitely many singularities and that they are all nondegenerate. Then,*

$$\sum_{P \in \text{Zeros } Z_\omega} \text{sgn}[-DZ(P)] = 1.$$

Proof. The proof has two steps. The first consists in constructing a specific vector field on S , which we call Z^Q , that has only one zero, it is inward-pointing along the boundary of S , and for which calculating the index is simple. The second step consists in showing that the excess demand function Z_ω is properly homotopic to the vector field Z^Q and that this proper homotopy is through ZPS vector fields.

For any fixed $Q \in C^{++}(M, \mathbb{R}^n)$ define the vector field $Z^Q : \bar{S} \rightarrow TS$ given by

$$Z^Q(P) = \left[\frac{Q(t)}{\langle P(t), Q(t) \rangle} \right] - P(t).$$

By construction, $Z^Q(P)$ has only one zero and is inward-pointing on the boundary. Its derivative $DZ_{(P)}^Q : T\bar{S} \rightarrow T(TS)$ is given by

$$DZ_{(P)}^Q(h) = -\frac{Q\langle h, Q \rangle}{\langle P, Q \rangle^2} - h$$

where

$$h \mapsto -\frac{Q\langle h, Q \rangle}{\langle P, Q \rangle^2}$$

is compact and

$$h \mapsto -h$$

is invertible; then $DZ^Q \in \mathcal{R}(T_P S)$.

Now let

$$-\frac{Q\langle h, Q \rangle}{\langle P, Q \rangle^2} - h = h'. \quad (4.3)$$

We need to solve for h . Then,

$$Q\langle h, Q \rangle + h\langle P, Q \rangle^2 = -h'\langle P, Q \rangle^2.$$

Acting Q on both sides we get,

$$\langle Q, Q \rangle \langle h, Q \rangle + \langle h, Q \rangle \langle P, Q \rangle^2 = -\langle h', Q \rangle \langle P, Q \rangle^2.$$

Solving for $\langle h, Q \rangle$ we get

$$\langle h, Q \rangle = \frac{-\langle h', Q \rangle \langle P, Q \rangle^2}{\langle Q, Q \rangle + \langle P, Q \rangle^2}$$

where the denominator never vanishes since $Q \in C^{++}(M, \mathbb{R}^n)$. Substituting $\langle h, Q \rangle$ in 4.3 we then get

$$h = h' + \frac{Q}{\langle P, Q \rangle^2} \left[\frac{\langle h', Q \rangle \langle P, Q \rangle^2}{\langle Q, Q \rangle + \langle P, Q \rangle^2} \right].$$

This shows that DZ^Q is invertible and therefore $DZ^Q \in G\mathcal{R}(T_P S)$. Furthermore, since it is not in the same component of the identity it has to be in $G\mathcal{R}^-(T_P S)$ and its only zero has index -1. The vector field Z^Q is inward pointing so reversing orientation will make it outward pointing with index of +1.

Up to this stage we have constructed a specific vector field on S , which we

called Z^Q , that has only one zero, it is inward-pointing along the boundary of S , and whose index we have shown to be $+1$. All that we need to do is to show that the excess demand function Z_ω is properly homotopic to the vector field Z^Q and that this proper homotopy is through ZPS vector fields.

Consider then the homotopy $F : S \times [0, 1] \rightarrow C(M, \mathbb{R}^n)$ given by

$$F(P, \alpha) = \alpha Z_\omega(P) + (1 - \alpha) Z^Q(P)$$

We have seen that

$$\begin{aligned} DZ_\omega(P) &= \sum_{i=1}^I \left\{ \lambda_i(P, w) (u_{xx}^i)^{-1} + \frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} Dw \right\} + \\ &+ \sum_{i=1}^I \left\{ -\frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} [f_i(P, w) + \lambda_i(P, w)(u_{xx}^i)^{-1} P] \right\} \end{aligned}$$

and

$$DZ^Q(P) = -\frac{Q \langle \cdot, Q \rangle}{\langle P, Q \rangle^2} - I$$

Hence

$$\begin{aligned}
DF(P, \alpha) &= \alpha DZ_\omega(P) + (1 - \alpha) DZ^Q(P) \\
&= \alpha \sum_{i=1}^I \left\{ \lambda_i(P, w) (u_{xx}^i)^{-1} + \frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} Dw \right\} + (1 - \alpha) \{-I\} \\
&\quad + \alpha \sum_{i=1}^I \left\{ -\frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} [f_i(P, w) + \lambda_i(P, w)(u_{xx}^i)^{-1} P] \right\} \\
&\quad + (1 - \alpha) \left\{ -\frac{Q \langle \cdot, Q \rangle}{\langle P, Q \rangle^2} \right\}
\end{aligned}$$

Finally notice that since $\alpha > 0$ and $1 - \alpha > 0$, then

$$\alpha \sum_{i=1}^I \left\{ \lambda_i(P, w) (u_{xx}^i)^{-1} + \frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} Dw \right\} + (1 - \alpha) \{-I\}$$

is invertible, and

$$\alpha \sum_{i=1}^I \left\{ -\frac{(u_{xx}^i)^{-1} P}{\langle P, (u_{xx}^i)^{-1} P \rangle} [f_i(P, w) + \lambda_i(P, w)(u_{xx}^i)^{-1} P] \right\} + (1 - \alpha) \left\{ -\frac{Q \langle \cdot, Q \rangle}{\langle P, Q \rangle^2} \right\}$$

has finite rank.

Hence, Z_ω is properly homotopic to the vector field Z^Q and that this proper homotopy is through ZPS vector fields.

□

4.8.2 Concluding remarks

We conclude from Proposition 51 that the number of equilibria of smooth infinite economies generically is odd. In particular, it can never be zero so this gives a new proof of existence.

Also, as a corollary of Proposition 51, we can also provide an infinite dimensional analogue of Dierker (1972); he shows

Theorem 52. *(Dierker, 1972) If the Jacobian of the excess supply function is positive at all Walras equilibria, then there is exactly one equilibrium.*

We show that:

Proposition 6. If the sign of the derivative of the excess supply function is positive at all Walras equilibria, i.e., when $-Z(P) = 0$ implies

$$D(-Z)(P) \in GR^+(T_P S)$$

then there is exactly one equilibrium.

Proof. The argument is straightforward. Since the index at each zero is > 0 and the sum of them is equal to 1, then there can only be one zero globally (and it has index = 1). □

Chapter 5

Global Uniqueness of Equilibria

5.1 Introduction

One of the first results that needs to be established in modeling the structure of markets is the existence of an equilibrium price system for an economy. The classical works of Arrow and Debreu have shown that all economies have at least one equilibrium although maybe not a unique one; an Edgeworth box can illustrate economies with a continuum of them. Debreu (1983) pointed out in his Nobel Prize Lecture that “the explanation of equilibrium given by a model of the economy would be complete if the equilibrium were unique, and the search for satisfactory conditions guaranteeing uniqueness has been actively pursued [...] However, the strength of the conditions that were proposed made it clear by the late sixties that global uniqueness was too demanding a requirement and that one would have to be satisfied with local uniqueness.”

Arrow and Hahn (1986, ch.9) have written a survey containing **sufficient con-**

ditions for a unique equilibrium. The most general sufficient condition known on the excess demand function, has been provided by Chichilnisky (1998) where she establishes that, under certain desirability conditions, if the reduced excess demand \hat{Z} has a nonvanishing Jacobian, then \hat{Z} is globally invertible. So in this case the market has a unique equilibrium.

However, necessary conditions are not yet fully understood. A **generic necessary condition** is given by the index theorem. Dierker (1972) shows that, under certain desirability assumptions, if the Jacobian of the reduced excess supply function is positive at all Walras equilibria, then there is exactly one equilibrium. Conversely, if there is only one Walras equilibrium and the economy is regular, then the index theorem also implies that the Jacobian evaluated at the equilibrium is positive. Thus, generically the condition is necessary and sufficient. An extension of these results has been given in the index theorem that we presented in our previous chapter for certain types of infinite economies.

In this chapter we present a “framework” for studying global uniqueness in a very general setting. We do answer the problem for infinite-dimensional consumption spaces, but we go further by also applying our method to finite dimensional economies with complete and incomplete markets.

To specify necessary and sufficient conditions we propose the following approach. The Walras correspondence assigns to every economy its set of equilibrium prices. Having exactly one equilibrium for each economy is the same as asking when the Walras correspondence is a differentiable bijection with a

differentiable inverse. However, working with functions is easier than with correspondences. We can instead ask for **necessary and sufficient** conditions that guarantee that the projection from the equilibrium manifold to the set of economies is a differentiable bijection and its inverse is also differentiable.

At the heart of our proof is a series of articles from Hadamard (1906a, 1906b, 1968) that asked the question: under which conditions is a local homeomorphism a global homeomorphism?

5.2 When a map is a diffeomorphism

We want to understand when the projection map from the equilibrium manifold to the set of economies is a differentiable bijection and its inverse is also differentiable. This is the notion of a diffeomorphism.

Definition 53. A differentiable map f is said to be a **diffeomorphism** if the map is injective and surjective and its inverse is also differentiable.

The Implicit Function Theorem will tell us that if the Jacobian of f is non-vanishing then f is a local homeomorphism. Of course this in itself, does not guarantee that f will be a bijection. The surprising fact is that if f is proper then it will be.

Definition 54. A continuous map f is said to be **proper** if $f^{-1}(K)$ is compact whenever K is compact.

We can ask for necessary and sufficient conditions that guarantee that f is a diffeomorphism between manifolds M and N . The answer to this question was

known to Hadamard; in a modern language it can be stated as:

Theorem 55. *(Hadamard, 1906a, 1906b, 1968) Let M and N be connected, oriented n -dimensional manifolds of class C^1 , without boundary, and suppose that N is simply connected. Then a C^1 map f from M to N is a diffeomorphism if and only if f is proper and the Jacobian of f never vanishes.*

Hadamard's result was rediscovered by Palais (1959 p.128-129) and Gordon (1972, 1973). There is a generalised version of Theorem 55 developed by Ho (1975) that does not require manifolds to be finite-dimensional, orientable or even boundaryless.

Theorem 56. *(Ho, 1975) Let M and N be connected manifolds of class C^1 and suppose that N is simply connected. Then a C^1 map f from M to N is a diffeomorphism if and only if f is proper and the Jacobian of f never vanishes.*

5.3 Complete Markets

5.3.1 The Market

As in previous chapters, consider an economy with L goods so that the consumption space is \mathbb{R}_{++}^L . Recall that prices are normalised so that they are in $S = \{p \in \mathbb{R}_{++}^L : \|p\| = 1\}$. There are $i = 1, \dots, I$ agents each of which is characterized by a smooth individual demand function $f_i : S \times (0, \infty) \rightarrow \mathbb{R}_{++}^L$ that satisfies Walras law, and by an initial endowment $\omega_i \in \mathbb{R}_{++}^L$. We suppose that an economy is parametrized by $\omega = (\omega_1, \dots, \omega_I) \in \Omega = X^I$.

The excess demand function $Z_\omega : S \rightarrow \mathbb{R}^L$ of an economy ω is given by

$$Z_{\omega}(P) = \sum_{i=1}^I [f_i(P, P \cdot \omega_i) - \omega_i]$$

We will also write the evaluation $Z(\omega, P) = Z_{\omega}(P)$. The reduced excess demand function $\hat{Z} : \Omega \times S \rightarrow \mathbb{R}^{L-1}$ is obtained from Z by deleting the last component.

5.3.2 The equilibrium manifold

Definition 57. The **equilibrium manifold** $\Gamma \subset \Omega \times S$ is defined by

$$\Gamma = \{(\omega, P) \in \Omega \times S : \hat{Z}(\omega, P) = 0\}.$$

Balasko (1988) shows that Γ is indeed a manifold and has dimension LI . Furthermore, he shows that it is contractible so it is connected and simply connected.

Definition 58. The natural **projection map** π is the projection $\Omega \times S \rightarrow \Omega$ restricted to the equilibrium manifold Γ . We simply write $\pi : \Gamma \rightarrow \Omega$.

Definition 59. An economy $\omega \in \Omega$ is **regular** (resp. **critical**) if and only if ω is a regular (resp. critical) value of the projection $\pi : \Gamma \rightarrow \Omega$.

5.3.3 Uniqueness of equilibria

We now give necessary and sufficient conditions on the projection map that guarantee global uniqueness of equilibria.

Theorem 60. *For every economy to have a unique equilibrium it is necessary and sufficient that (i) there are no critical economies and (ii) every compact set of economies has a compact set of equilibrium prices.*

Theorem 60 could be rephrased in the following way.

Theorem 61. *The projection map $\pi : \Gamma \rightarrow \Omega$ is a diffeomorphism if and only if π is proper and the Jacobian of π never vanishes.*

At this point it is worth mentioning that a similar result to theorem 60 has been known to Balasko (1988). That is, he shows that under suitable desirability assumptions, the projection map $\pi : \Gamma \rightarrow \Omega$ is proper. Hence there are no critical economies if and only if every economy has a unique equilibrium. In a way our result is that “properness” is not just a suitable assumption but it is crucial in order to extend our results to other, more general, settings ¹.

Theorems 60 and 61 are an application of the ideas of Hadamard, Palais, Gordon and Ho, and the proof presents no new mathematical insight. However, we reproduce it with an economic interpretation for completeness.

Proof of Theorem 60. By a result of Balasko (1988), the equilibrium manifold Γ is connected. Also notice that Ω is simply connected.

¹There is another way of seeing this which I learned from Yves Balasko through private communication (2007). Basically, global uniqueness is equivalent to having the set of endowments coinciding with the component of the set of regular economies that contains the set of Pareto optima. We do not develop further these ideas in the thesis since we have not defined the concept of Pareto optimality for infinite economies.

First suppose that $\pi : \Gamma \rightarrow \Omega$ is a diffeomorphism. Then the Jacobian of π will never vanish; so every economy is regular. But also, by assumption, the Walras correspondence π^{-1} is a continuous map so it must map a set of compact economies to a set of compact equilibrium prices. Hence π is proper.

Conversely, now suppose that (i) there are no critical economies and (ii) every compact set of economies has a compact set of equilibrium prices. We want to show that π is a diffeomorphism. Since there are no critical economies, the Implicit Function Theorem guarantees that the inverse is differentiable. All we need to show then is that π is a bijection.

Since π is proper, we can use a result of Palais (1970) that a proper map sends closed sets into closed sets, i.e. $\pi(\Gamma)$ is closed. But also, since there are no critical economies, π is a local homeomorphism so it also sends open sets to open sets, i.e. $\pi(\Gamma)$ is open. Hence $\pi(\Gamma)$ is an open, closed and nonempty subset of Ω . So $\pi(\Gamma) = \Omega$. This shows that π is surjective.

We now show that π is injective. Consider two points γ_1, γ_2 in the equilibrium manifold Γ such that $\pi(\gamma_1) = \pi(\gamma_2) = \omega$. Since Γ is connected, we can consider a path $\alpha(t)$ in Γ connecting γ_1 to γ_2 . Then $\pi \circ \alpha(t)$ is a loop in Ω based in ω . We also know that Ω is simply connected, so we may use a homotopy $F(s, t)$ such that $F(0, t) = \pi \circ \alpha(t)$ and $F(1, t) = \omega$. Since we have seen that π is surjective, proper and a local homeomorphism from Γ to Ω , then π must be a covering projection. And every covering projection has the homotopy lifting property (Hatcher, 2002, p.60). So there has to be a unique lifting $\tilde{F}(s, t)$ of

$F(s, t)$ with $\tilde{F}(0, t) = \alpha(t)$. The lift of $\tilde{F}(1, t)$ must be a connected set containing both γ_1 and γ_2 . But $\pi^{-1}(\omega)$ is discrete, so $\gamma_1 = \gamma_2$. \square

5.3.4 Invertibility of the excess demand function

Using the same arguments as in Theorem 60, we can provide a converse to Chichilnisky (1998). We only need to notice that S is connected and \mathbb{R}^{L-1} is simply connected. Hence we have:

Theorem 62. *The reduced excess demand function $\hat{Z}_\omega : S \rightarrow \mathbb{R}^{L-1}$ of an economy ω is globally invertible if and only if \hat{Z}_ω has a nonvanishing Jacobian and it is a proper map.*

Proof. First note that S is the positive orthant of the unit sphere in \mathbb{R}_{++}^L and so it is connected. Also, \mathbb{R}^L is obviously simply connected.

Now, suppose that $\hat{Z}_\omega : S \rightarrow \mathbb{R}^{L-1}$ is a diffeomorphism. Then the Jacobian of \hat{Z} will always be different from zero. Additionally, since \hat{Z} is a diffeomorphism, its inverse \hat{Z}^{-1} is continuous and so must map closed sets into closed sets. Therefore, \hat{Z} is a proper map.

Conversely, assume now that \hat{Z}_ω has a nonvanishing Jacobian and it is a proper map. The implicit function theorem guarantees that everywhere the inverse is differentiable. It only remains to be shown that it is a bijection.

Palais (1970) shows that a proper map send closed sets into closed sets and so $\hat{Z}(S)$ is closed. But also, since the Jacobian of \hat{Z} never vanishes, it is a local

homeomorphism so it also sends open sets onto open sets so that $\hat{Z}(S)$ is also open. Since $\hat{Z}(S)$ is an open, close and nonempty subset of \mathbb{R}^L it must be that $\hat{Z}(S) = \mathbb{R}^L$ and so it is surjective.

Finally, consider two price systems p_1 and p_2 in S such that $\hat{Z}(p_1) = \hat{Z}(p_2) = x$. Since S is topologically a disk, there is a path $\alpha(t)$ connecting p_1 and p_2 in S . And so $\hat{Z} \circ \alpha(t)$ is a loop in \mathbb{R}^l based in x . We may use a homotopy $F(s, t)$ such that $F(0, t) = \hat{Z} \circ \alpha(t)$ and $F(1, t) = x$. Since we have seen that \hat{Z} is surjective, proper and a local homeomorphism from S to \mathbb{R}^l , then by a result of Ho (1975, p.239), \hat{Z} must be a covering projection. And every covering projection has the homotopy lifting property property (Hatcher, 2002, p.60). So there has to be a unique lifting $\tilde{F}(s, t)$ of $F(s, t)$ with $\tilde{F}(0, t) = \alpha(t)$. The lift of $\tilde{F}(1, t)$ must be a connected set containing both p_1 and p_2 . But $\hat{Z}^{-1}(x)$ is discrete, so $p_1 = p_2$. \square

5.4 Incomplete Markets

5.4.1 The Market

For a complete survey of general equilibrium with incomplete markets, we refer the reader to Magill and Shafer (1991).

Consider an economy where there are two time periods $t = 0, 1$. At $t = 0$ there are spot markets for commodities and security markets for assets that pay bundles of commodities at $t = 1$, the bundle paid depending on the state of nature. At $t = 1$ agents cash in their portfolios of assets and their endowments,

trading the proceeds on spot markets for commodities. There are L commodity types, n states of nature at $t = 1$ and k assets.

The consumption space is $X = \mathbb{R}_{++}^{L(n+1)}$ and a price p is in X . Endowments are in $\Omega = X^I$. Let $G_{k,n}$ denote the Grassmanian manifold.

For the precise definitions we refer the reader to Duffie and Shafer (1985).

For each agent $i > 2$ define its individual demand function $f_i : X \times G_{k,n} \times X \rightarrow X$ and let the excess demand function $Z : X \times G_{k,n} \times \Omega \rightarrow X$ be given by

$$Z(p, L, \omega) = f_1(p, 1) + \sum_{i=1}^I f_i(p, L, \omega_i) - \sum_{i=1}^I \omega_i$$

5.4.2 The Pseudo-Equilibrium Manifold

Define the **pseudo-equilibrium manifold** \mathcal{E} to be

$$\mathcal{E} = \{(p, L, \omega, a) : Z(p, L, \omega) = 0, K_\sigma(p, L, a) = 0\}$$

with $K_\sigma : X \times G_{k,n} \times \mathbb{R}^{Lnk}$ as given in Duffie and Shafer (1985, p. 294).

Intuitively, $K_\sigma(p, L, a) = [I|\phi_\sigma(L)] P_\sigma V(p, a)$ where σ is a permutation of $\{1, 2, \dots, n\}$ and if $L \in G_{k,n}$ we can write it in canonical form $[I|E] P_\sigma \in L$ where E is an $(n - k) \times k$ matrix.

Duffie and Shafer (1985) show that \mathcal{E} is indeed a submanifold of $X \times G_{k,n} \times$

$\Omega \times \mathbb{R}^{Lnk}$ with empty boundary and of dimension $mL(n+1) + Lnk$.

The structure of the pseudo-equilibrium manifold \mathcal{E} is a subject that was explored in detail by Chichilnisky and Heal (1996). In their paper they show that

Theorem 63. *(Chichilnisky and Heal, 1996) The pseudo-equilibrium manifold \mathcal{E} is a covering space for a fiber bundle over the Grassmannian manifold $G_{k,n}$. The manifold \mathcal{E} is topologically equivalent to either the Grassmannian manifold $G_{k,n}$ or to the manifold of oriented k -subspaces in \mathbb{R}^n . In general it is not contractible when the market is incomplete, i.e. when $k < n$ and $n > 2$. However, when the market is complete, i.e. when $k = n$, then the pseudo-equilibrium manifold is $\mathcal{E} = G_{n,n}$ and therefore contractible.*

The manifold \mathcal{E} is not contractible, so we need to study its connectedness. Recall (Hatcher, 2002) that:

$$\pi_1(G_{k,n}) = \begin{cases} \mathbb{Z}_2, & \text{for all } (k,n) \neq (1,2) \\ \mathbb{Z}, & \text{if } (k,n) = (1,2) \end{cases}$$

and so

$$\pi_1(\mathcal{E}) = \begin{cases} \mathbb{Z}_2, & \text{for all } (k,n) \neq (1,2) \\ \mathbb{Z}, & \text{if } (k,n) = (1,2) \end{cases}$$

which means that \mathcal{E} is connected and additionally $\Omega \times \mathbb{R}^{Lnk}$ are simply connected.

We denote the projection map π as the projection $X \times G_{k,n} \times \Omega \times \mathbb{R}^{Lnk} \rightarrow \Omega \times \mathbb{R}^{Lnk}$ restricted to the pseudo-equilibrium manifold \mathcal{E} . We simply write $\pi : \mathcal{E} \rightarrow \Omega \times \mathbb{R}^{Lnk}$.

An economy $(\omega, a) \in \Omega \times \mathbb{R}^{Lnk}$ is regular (resp. critical) if and only if (ω, a) is a regular (resp. critical) value of the projection $\pi : \mathcal{E} \rightarrow \Omega \times \mathbb{R}^{Lnk}$.

5.4.3 Global Uniqueness of Equilibria for GEI

Using once again Hadamard's results we can prove theorem 64.

Theorem 64. *For every economy and asset structure with incomplete financial markets to have a unique equilibrium it is necessary and sufficient that (i) there are no critical economies and (ii) every compact set of economies and asset structures has a compact set of pseudo-equilibria.*

Proof. We only need to consider that Chichilnisky and Heal's result imply that

$$\pi_1(\mathcal{E}) = \begin{cases} \mathbb{Z}_2, & \text{for all } (k, n) \neq (1, 2) \\ \mathbb{Z}, & \text{if } (k, n) = (1, 2) \end{cases}$$

and so \mathcal{E} is connected. The proof follows line by line that of theorem 60. \square

5.5 Infinite Economies

In order to study global uniqueness of equilibria for smooth infinite economies, it is necessary to establish the topology of the infinite equilibrium manifold. It

turns out that its topology can be studied in a similar way to Balasko (1975). Almost line by line our results will be an extension. The reason is that we only need to see that the equilibrium manifold is made of linear fibers given by the equations that define it.

Consider the map $f : S \times (\mathbb{R}_{++})^I \rightarrow S \times \Omega$ given by

$$f(P, w_1, \dots, w_I) = (P, f_1(P, w_1), \dots, f_I(P, w_I))$$

and the map $\phi : S \times \Omega \rightarrow S \times \mathbb{R}^I$ given by

$$\phi(P, \omega_1, \dots, \omega_I) = (P, P \cdot \omega_1, \dots, P \cdot \omega_I)$$

Then the map $f : S \times (\mathbb{R}_{++})^I \rightarrow S \times \Omega$ is a cross-section in bundle $(\phi, S \times (\mathbb{R}_{++})^I, S \times \Omega)$.

Theorem 65. *The infinite equilibrium manifold is contractible. In particular it is arc-connected and simply connected.*

Proof. The proof follows line by line the proof of theorem 1 in (Balasko, 1975). \square

Theorem 66. *For every smooth infinite economy to have a unique equilibrium it is necessary and sufficient that (i) there are no critical economies and (ii) every compact set of economies has a compact set of equilibrium prices.*

Proof. From theorem 65 the infinite equilibrium manifold Γ is connected. Also notice that Ω is simply connected as it simply is an open neighborhood of cross products of $C^{++}(M, \mathbb{R}^n)$.

First suppose that $\pi : \Gamma \rightarrow \Omega$ is a diffeomorphism. Then the Frechet derivative of π is surjective everywhere; so every infinite economy is regular. But also, by assumption, the Walras correspondence π^{-1} is a continuous map so it must map a set of compact economies to a set of compact equilibrium prices. Hence π is proper.

Conversely, now suppose that (i) there are no critical economies and (ii) every compact set of economies has a compact set of equilibrium prices. We want to show that π is a diffeomorphism. Since there are no critical economies, the Implicit Function Theorem between Banach spaces guarantees that the inverse is Frechet differentiable. All we need to show then is that π is a bijection.

Since π is proper, we can use a result of Palais (1970) that a proper map sends closed sets into closed sets, i.e. $\pi(\Gamma)$ is closed. But also, since there are no critical economies, π is a local homeomorphism so it also sends open sets to open sets, i.e. $\pi(\Gamma)$ is open. Hence $\pi(\Gamma)$ is an open, closed and nonempty subset of Ω . So $\pi(\Gamma) = \Omega$. This shows that π is surjective.

We now show that π is injective. Consider two points γ_1, γ_2 in the equilibrium manifold Γ such that $\pi(\gamma_1) = \pi(\gamma_2) = \omega$. Since Γ is connected, we can consider a path $\alpha(t)$ in Γ connecting γ_1 to γ_2 . Then $\pi \circ \alpha(t)$ is a loop in Ω based in ω . We also know that Ω is simply connected, so we may use a homotopy $F(s, t)$ such that $F(0, t) = \pi \circ \alpha(t)$ and $F(1, t) = \omega$. Since we have seen that π is surjective, proper and a local homeomorphism from Γ to Ω , then π must be a covering projection. And every covering projection has the homotopy lifting

property property (Hatcher, 2002, p.60). So there has to be a unique lifting $\tilde{F}(s, t)$ of $F(s, t)$ with $\tilde{F}(0, t) = \alpha(t)$. The lift of $\tilde{F}(1, t)$ must be a connected set containing both γ_1 and γ_2 . But $\pi^{-1}(\omega)$ is discrete, so $\gamma_1 = \gamma_2$. \square

5.6 Critical economies and the number of equilibria

In our previous section, we studied the natural projection map as a covering map over the set of economies. This section studies the effect that the very existence of critical economies has on global properties of the projection map. This section is based on the general results obtained by Plastock (1978).

Let $B \subset \Gamma$ denote the set of critical prices and $\Sigma \subset \Omega$ be the set of critical economies. It is worth mentioning that theorem 66 could be rephrased as follows.

Theorem 67. *In order for every smooth infinite economy to have a unique equilibrium it is necessary and sufficient that (i) every compact set of economies has a compact set of equilibrium prices, (ii) the projection map $\pi : \Gamma \rightarrow \Omega$ is a Fredholm map of index zero and (iii) $B = \emptyset$.*

We can then ask how the set of critical economies affects the uniqueness of equilibria. We first calculate the Fredholm index of the projection map.

Theorem 68. *The map $\pi : \Gamma \rightarrow \Omega$ is Fredholm of index zero.*

Proof. The projection $\pi : \Gamma \rightarrow \Omega$ is Fredholm by a simple application of Abraham and Robbin (1967, p.48). The index is constant across Γ since we have shown that it is connected. \square

Theorem 69. *Let Σ denote the singular set in Ω . Then $\pi : \Gamma - \pi^{-1}(\Sigma) \rightarrow \Omega - \Sigma$ is a covering space map.*

Proof. Since π is proper then by Palais (1970) it sends closed sets to closed sets. Hence the set of critical economies $\pi(B) = \Sigma$ is closed and therefore $\Gamma - \pi^{-1}(\Sigma)$ and $\Omega - \Sigma$ are both open.

The idea now is to show that every compact set of regular economies has a compact set of equilibrium prices. That is, to show that $\pi : \Gamma - \pi^{-1}(\Sigma) \rightarrow \Omega - \Sigma$ is a proper map. If this is the case, π will be both an open and a closed map and hence surjective. Hence, it will be a covering space map.

Let K be compact subset of $\Omega - \Sigma$. Then K is also compact in Ω . But $\pi^{-1}(K) \subset \Gamma - \pi^{-1}(\Sigma)$ and so it is compact in this set. \square

The previous result shows that if Σ is small enough, the surjectivity of π is guaranteed and so we can focus on the study of π as a covering map and its behaviour on the critical set. It turns out that critical prices are removable in the following sense.

Theorem 70. *Let B be the set of critical equilibrium prices in Γ . Then, isolated critical prices are removable, i.e., if $p \in B$ is isolated in B then π is a local homeomorphism about p .*

Proof. The proof is Theorem 4 of Plastock (1978). \square

Our previous result shows that at isolated critical prices, locally we have $\pi^{-1}(\pi(B)) = B$. Our final result shows a globalisation of this result.

Theorem 71. *If the set of critical prices B is the countable union of compact sets, then $\pi^{-1}(\pi(B)) = B$ and π is a global diffeomorphism of $\Gamma - B$ onto $\Omega - \Sigma$.*

Proof. The proof is Theorem 6 of Plastock (1978). □

Appendix A

Analytic Preliminaries

This chapter reviews some of the analytic facts that will be needed throughout the dissertation.

We start in section A.1 by reviewing concepts from linear analysis mainly the notion of a Fredholm operator. This material can be studied in (Lax, 2002) or (Berger, 1977). In section A.2 we then survey some notions of nonlinear analysis, primarily the notion of a nonlinear Fredholm operator: this concept is central to the results that we obtain through the dissertation and the material can be found in (Berger, 1977) or the seminal paper of Smale (1965).

A.1 Facts from functional analysis

A.1.1 Banach spaces

Banach spaces

A Banach space $(X, \|\cdot\|)$ is a normed vector space (over the real numbers throughout) that is complete with respect to the metric $d(x, y) = \|x - y\|$.

Hilbert spaces

A Hilbert space H is a vector space with a positive-definite inner product $\langle \cdot, \cdot \rangle$ that defines a Banach space upon setting $\|x\|^2 = \langle x, x \rangle$ for $x \in H$.

Subspaces

Closed linear subspaces of Banach (Hilbert) spaces are again Banach (Hilbert) spaces. As Berger (1997) pointed out, the rather intricate geometry of general Banach spaces can be seen by noting that there are closed subspaces M of a Banach space X for which no closed subspace N of X exists satisfying $X = M \oplus N$. Fortunately, this does not occur if

- X is a Hilbert space,
- $\dim M < \infty$, or
- $\operatorname{codim} M < \infty$

Bounded linear functionals and the conjugate space

A bounded linear functional $h(x)$ defined on a Banach space X is a linear mapping $X \rightarrow \mathbb{R}$ such that $|h(x)| \leq K\|x\|_X$ for some constant K independent of $x \in X$.

The set of all bounded linear functionals on X , denoted X^* , is called the conjugate space of X . It is a Banach space with respect to the norm $\|h\| = \sup|h(x)|$ over the sphere $\|x\|_X = 1$. If $(X^*)^* = X$, then the space X is called reflexive.

Compactness

One says that a set M of a Banach space X is compact if M is closed (in the norm topology) and such that every sequence in M contains a strongly convergent subsequence.

Bounded linear operators

A linear operator L with domain X and range contained in Y , (X, Y Banach spaces) is bounded if there is a constant K independent of $x \in X$ such that $\|Lx\|_Y \leq K\|x\|_X$ for all $x \in X$. The set of such maps for fixed X, Y is again a Banach space, denoted $L(X, Y)$ with respect to the norm $\|L\| = \sup\|Lx\|_Y$ for $\|x\|_X = 1$.

Adjoint operators

Any bounded linear operator $L \in L(X, Y)$ has an adjoint $L^* \in L(Y^*, X^*)$ that is uniquely defined by setting $L^*g = f$ where $f(x) = g(Lx)$ for every bounded linear functional $f \in X^*$. Thus $\|L^*\| = \|L\|$; and for two operators $L_1, L_2 \in L(X, Y)$, $(\alpha L_1 + \beta L_2)^* = \alpha L_1^* + \beta L_2^*$ and $(L_1 L_2)^* = L_2^* L_1^*$.

Compact linear operators

A linear operator $C \in L(X, Y)$ is called **compact** if for any bounded set $B \subset X$, $C(B)$ is conditionally compact in Y . Bounded linear mappings with finite-

dimensional ranges are automatically compact; and conversely, if X and Y are Hilbert spaces, then a compact linear mapping C is the uniform limit of such mappings.

Compact perturbations of the identity

Let $C \in L(X, X)$ be compact, and set $L = I + C$. Then

1. L has closed range
2. $\dim \ker L = \dim \operatorname{coker} L < \infty$
3. there is a finite integer β such that $X = \ker(L^\beta) \oplus \operatorname{range}(L^\beta)$ and L is a linear homeomorphism of $\operatorname{range}(L^\beta)$ onto itself

A.1.2 Fredholm operators and their generalizations

Definitions

An operator $L \in L(X, Y)$ is called **Fredholm** if:

1. the range of L is closed in Y , and
2. the subspaces $\ker L$ and $\operatorname{coker} L$ are finite dimensional

The set of Fredholm maps contained in $L(X, Y)$ is denoted by $\Phi(X, Y)$. It can be shown that $\Phi(X, Y)$ is an open subset of $L(X, Y)$. The index of a Fredholm map L , $\operatorname{ind} L \in \mathbb{Z}$, can be defined by either of the formulas

$$\operatorname{ind} L = \dim \ker L - \dim \operatorname{coker} L = \dim \ker L - \dim \ker L^* \quad (\text{A.1})$$

Properties

Invariance of index under compact perturbations It can be shown that the index is invariant under compact perturbations and perturbations by elements of $L(X, Y)$ of sufficiently small norm. Thus, the index is constant on the connected components of $\Phi(X, Y)$. Moreover, if $A \in \Phi(X, Y)$ and $B \in \Phi(Y, Z)$, then $BA \in \Phi(X, Z)$ and $\text{ind}BA = \text{ind}B + \text{ind}A$. The subset of Fredholm maps of index k is denoted by $\Phi_k(X, Y)$.

Compact perturbations of the identity Compact perturbations of the identity are Fredholm operators of index zero, and conversely, any Fredholm map $L \in \Phi_0(X, Y)$ differs from a compact perturbation of the identity only by a linear homeomorphism in $L(X, Y)$.

Semi-Fredholm operators

An operator L is called **semi-Fredholm** if L has closed range but $\dim \text{coker} L = \infty$. The set of semi-Fredholm operators is denoted by $\Phi_+(X, Y)$.

A.2 Nonlinear operators

A.2.1 Basic definitions and properties

Continuity

A map f from X to Y is called continuous (with respect to convergence in norm) if $x_n \rightarrow x$ in X always implies $f(x_n) \rightarrow f(x)$ in Y . The set of continuous mappings will be denoted by $C^0(X, Y)$.

Boundedness

The map f is said to be bounded if it maps bounded sets into bounded sets. It is called locally bounded if each point in the domain of f has a bounded neighbourhood N such that $f(N)$ is bounded. Continuous maps of a finite-dimensional Banach space X into a Banach space Y are necessarily bounded.

Differentiation

Let f be a map from X to Y . We say that f is Fréchet differentiable at x_0 if there is a linear operator $A \in L(X, Y)$ such that in a neighbourhood U of x_0 ,

$$\|f(x) - f(x_0) - A(x - x_0)\| = o(\|x - x_0\|) \quad (\text{A.2})$$

In this case we write $A = f'(x_0)$ and $f'(x_0)$ is called the Fréchet derivative of f at x_0 . If the mapping $x \rightarrow f'(x)$ of $X \rightarrow L(X, Y)$ is continuous at x_0 , f is called C^1 at x_0 .

Compactness

Let X and Y be Banach spaces and U be a subset of X . Then a map f from U to Y is compact if f is continuous and maps bounded subsets of U into conditionally compact subsets of Y , and we write $f \in K(U, Y)$. All continuous, bounded mappings of a subset U of a Banach space X into a finite-dimensional Banach space Y are compact.

A.2.2 Nonlinear Fredholm operators

Definitions and properties

Definition Let X, Y be Banach spaces and U a connected open subset of X . A mapping $f \in C^1(U, Y)$ is called a **nonlinear Fredholm operator** if the Fréchet derivative of f , $f'(x)$ is a linear Fredholm map $\in L(X, Y)$ for each $x \in U$. In this case, the **index** of f , $\text{ind} f$, is defined by setting $\text{ind} f = \text{ind} f'(x) = \dim \ker f'(x) - \dim \text{coker} f'(x)$ for $x \in U$.

Constant The index of f is independent of the choice of $x \in U$. This is because $\text{ind} f : U \rightarrow \mathbb{Z}$ is continuous; and since U is connected, $x \in U$ implies $\text{ind} f(x)$ is constant.

Examples Some examples of Fredholm maps and their indices.

- Any smooth map between finite-dimensional Banach spaces is a Fredholm map.
- Any diffeomorphism between Banach spaces is a Fredholm map of index zero.
- If $f(x)$ is any Fredholm map and $C(x) \in C^1(U, Y)$ is a compact operator, then $f + C$ is a Fredholm operator and $\text{ind}(f + C) = \text{ind} f$.

Singularities

Regular and critical points Let $f \in C^1(U, Y)$, then $x \in U$ is a regular point for f if $f'(x)$ is a surjective linear mapping in $L(X, Y)$. If $x \in U$ is not regular, x is called singular.

Regular and critical values Similarly, singular and regular values y of f are defined by considering the sets $f^{-1}(y)$. If $f^{-1}(y)$ has a singular point, y is called a singular value, otherwise y is a regular value.

Basic properties

- The singular points of a Fredholm operator $f \in C^1(X, Y)$ are closed.

A.2.3 Proper mappings

Definitions

An operator $f \in C^0(X, Y)$ is said to be proper if the inverse image of any compact set C in Y , $f^{-1}(C)$ is compact in X . The importance of this notion resides in the fact that the properness of an operator f restricts the size of the solution set $S_p = \{x : x \in X, f(x) = p\}$ for any fixed $p \in Y$.

Equivalent definitions

Theorem 72. (Berger) *Let $f \in C^0(X, Y)$, then the following statements are equivalent:*

- (i) f is proper
- (ii) f is a closed mapping and the solution set $\{x : x \in X, f(x) = p\}$ is compact for any fixed $p \in Y$
- (iii) if X and Y are finite dimensional, then f is coercive (in the sense that $\|f(x)\| \rightarrow \infty$ whenever $\|x\| \rightarrow \infty$).

Stability of the solution set

Theorem 73. (Berger) Let $f \in C^0(X, Y)$ be proper. Then:

(i) for every $p \in Y$ and every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|f(x) - p\| \leq \delta \text{ implies } \|x - f^{-1}(p)\| \leq \epsilon$$

(ii) if $g \in C^0(X, Y)$, then $\|f(x) - g(x)\| \leq \delta$ for all $x \in X$ implies that $d(S_p(f), S_p(g)) \leq \epsilon$.

Constant number of solutions

Theorem 74. (Berger) Let X and Y be Banach spaces and $f \in C^0(X, Y)$. Suppose U and V are open subsets of X and Y , respectively, such that f maps U onto V , is locally invertible, and proper on U . Then the function $c_p =$ the number of points in $S_p(U) = \{x : x \in U, f(x) = p\}$ is finite and constant in each component of V .

Singularity

Theorem 75. (Berger) If $f \in C^1(X, Y)$ is a proper Fredholm operator of index zero and S denotes the singular set of f , then c_y is constant on every (connected) component of $Y - f(S)$. More generally for proper operators of higher index the sets $f^{-1}(y)$ are homeomorphic.

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